Strong boundedness and distortion in transformation groups

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Theorem (Higman–Neumann–Neumann) \( \Gamma \) countable group. There is a finitely generated group \( H \) (generated by two elements) with \( \Gamma \subset H \).
Higman embedding theorem: $\Gamma$ countable group. There is a finitely generated group $H$ with $\Gamma \subset H$.

Relative version: Fix an (uncountable) group $G$. (e.g. Lie group, automorphism group, homeomorphism group) Let $\Gamma \subset G$ be countable. Is there a finitely generated $H \subset G$ with $\Gamma \subset H$?

Examples of $G$ where the answer is always positive?
PROBLEM 111: SCHREIER

Does there exist an uncountable group with the property that every countable sequence of elements of this group is contained in a subgroup which has a finite number of generators? In particular, do the groups $S_\infty$ and the group of all homeomorphisms of the interval have this property?

(circa 1935)

Ory istnieje grupa nieprzeliczalna o tej własności, że każdy ciąg przeliczalny elementów tej grupy zawarty jest w podgrupie o skończonej ilości troszeczek.

W szczególności, ozy widoczności tej posiadają grupę poza własną grupą homeomorfizmem podstawki.
Theorem (implicit in Sabbagh (1975) using Scott (1951))
Groups with this property exist.

Theorem (Galvin, 1995)
\( S_\infty \) has this property.

Theorem (Le Roux – M., 2016)
Homeo(I) has this property.
So does Homeo_0(M) and Diff_0(M) for any manifold M.

...and in many cases, this is a consequence of something stronger.
Related concepts

Serre’s (FA):
i) G doesn’t split,  ii) doesn’t have a $\mathbb{Z}$ quotient, and
iii) G finitely generated.
Related concepts

Serre’s (FA):

i) $G$ doesn’t split,  
ii) doesn’t have a $\mathbb{Z}$ quotient, and  
iii) $G$ not a countable increasing union of proper subgroups

implied by Schreier’s property

(proof of contrapositive)

Suppose $G = \bigcup G_n$, with $G_0 \varsubsetneq G_1 \varsubsetneq G_2 \varsubsetneq \ldots$ 
choose $g_i \in G_{i+1} \setminus G_i$.

$\{g_n\}$ not in any f.g. subgroup. □

First examples by Koppelberg–Tits ’74.

Many recent examples, starting with Bergman ’04.
Related concepts

Definition
$G$ has **strong boundedness** if every length function $G \to [0, \infty)$ is bounded.

\[ \ell(g^{-1}) = \ell(g), \ \ell(id) = 0, \ \ell(gh) \leq \ell(g) + \ell(h) \]

\[\iff\] any isometric action of $G$ on a metric space has bounded orbits.

Definition
$G$ has **strong distortion** if $\exists M$ and sequence $w_n \to \infty$ such that, for any $\{g_n\} \subset G$, have $M$-element set $S \subset G$ generating this sequence, and $|g_n|_S < w_n$.

\[\Rightarrow\] subgroup distortion

Strong distortion $\Rightarrow$ strong boundedness $\nRightarrow$ (Cornulier)

\[\Rightarrow\] Schreier’s property

For examples see [1], [4]
Distortion in transformation groups

Distortion of $G \subset \text{Diff}(M) \leftrightarrow \text{dynamics of action of } G$. 
Distortion in transformation groups

- **fixed points** (Franks–Handel 2006):
  \( f \in \text{Diff}(\Sigma) \), preserves \( \mu \), and distorted in some f.g. subgroup.
  Then \( \text{supp}(\mu) \subset \text{fix}(f) \).

- **growth of derivatives** (Calegari–Freedman)
  \( f(x) = x, \quad \|Df_x\| > 0 \) i.e. has eigenvalue of norm \( \neq 1 \)
  \( \Rightarrow \langle f \rangle \) not distorted in any f.g. subgroup. of \( \text{Diff}(M) \), \( M \) compact
  \[ \ell(f) := \|Df\| \] is unbounded length function on \( \text{Diff}(M) \)

- **“stretch”**
  \[ \ell(f) := \sup_{x,y \in \tilde{M}} |d(\tilde{f}(x), \tilde{f}(y)) - d(x, y)| \]
  unbounded on \( \text{Homeo}(M) \) if \( \pi_1(M) \) infinite.

- **Much more…** Polterovich [11], Hurtado [8], Gromov [5], etc.
Our results

**Theorem (Le Roux – M.)**

\[ \text{Diff}^r(\mathbb{R}^n) \text{ has strong distortion. } r \neq n + 1 \]

*Given sequence \( g_n \), can build \( S \) with \( |S| = 17 \) and \( |g_n|_S \leq 50n + 24 \). Surely non optimal!*

**Open question:** Minimum \( |S| \)? Obstruction to \( |S| = 2 \)?

Theorem false for:
- \( r \geq 1, M \text{ compact}, \)
- \( r = 0, \pi_1(M) \text{ infinite} \)
  but...
Our results

Theorem (Le Roux – M.)

*M compact, or homeomorphic to interior of compact manifold.* Given \( g_n \subset \text{Diff}^r(M) \), can build finite \( S \) that generates all \( g_n \) (no control on word length).

Consequences:

1. Can do G.G.T. for countable subgroups (by including in f.g. subgroup).

   Remark: G.G.T. / large-scale geometric concepts make sense in Homeo_0(M) by [Mann-Rosendal 15]

2. These groups are all topologically f.g., have countable dense subgroup.

3. “natural” examples of groups with Schreier’s property
Proving strong distortion

Easy case: \( G = \text{Diff}^0(\mathbb{R}) \). (\( \cong \text{Homeo}(I) \), Schrier's question)

Classical trick (G. Fisher '60)
\( \{f_n\} \) supported in \( K \text{ compact} \subset \mathbb{R} \Rightarrow \) generated by \( s, t, F \).

Proof:
Define \( t \) and \( s \)

Now define \( F \) consisting of (conjugate) copies of \( f_n \)
\[ F = \prod_{j,n \geq 0} s^n t^j f_n t^{-j} s^{-n} \]

Check: \( F \circ (tF^{-1}t^{-1}) = f_0. \)

Make \( f_n \) by conjugating by power of \( s \) first: \( f_n = [F^{s-n}, t] \)
Proving strong distortion

Lemma
Given \( \{f_n\} \) sequence in \( \text{Homeo}_+ (\mathbb{R}) \). Can find \( X, Y \subset \mathbb{R} \) (unions of disjoint intervals) such that

\[
f_n = g_n h_n k_n
\]

\( \text{supp}(g_n) \subset X, \text{supp}(h_n) \subset Y \) and \( \text{supp}(k_n) \) compact.

idea of proof: \( I \cup f(I) \subset J \Rightarrow \) can find \( g \) agreeing with \( f \) on \( I \) and identity outside neighborhood of \( J \).
Proving strong distortion

Lemma
Given \( \{f_n\} \) sequence. Can find \( X, Y \subset \mathbb{R} \) (unions of disjoint intervals) such that
\[ f_n = g_n h_n k_n \]
\[
\text{supp}(g_n) \subset X, \text{supp}(h_n) \subset Y \text{ and sup}(k_n) \text{ compact.}
\]

Now use classical trick three times:
1. Find \( d \) so that \( d^n k_n d^{-n} \) supported in \([0, 1]\). \( \rightsquigarrow \) classical trick. \[
|d^n k_n d^{-n}|_S \leq 4n + 4.
\]
2. Trick works for \( X \) rather than \( K \), so \( g_n = [G^{s^{-n}}, t] \) for some \( s, t, G \), \( \Rightarrow g_n \) has length \( 4n + 4 \) in finite set.

3. Same for \( Y, h_n \).

\(\square\) distortion for Homeo_0(\(\mathbb{R}\))
Other (less easy) cases

- Homeo_0(\mathbb{R}^n), Diff_0(\mathbb{R}^n) can also do \( f_n = g_n h_n k_n \)

...but need to use annulus theorem in higher dimensions.

- More difficulty for Diff_0(\mathbb{R}^n)
  (classical trick doesn’t produce diffeomorphisms)
Diff\(_r^0(\mathbb{R}^n)\) and Diff\(_r^0(M)\) case

Tools:
- Simplicity of Diff\(_r^c(M)\), \(r \neq \dim(M) + 1\).
- Fragmentation: \(f\) close to identity \(\Rightarrow\) product of diffeomorphisms supported on elements of open cover

Related: *Fragmentation norm*

- Replacement of classical trick using idea inspired by Avila
  - Construction of Burago–Ivanov–Polterovich
  - Technical work
Open questions:

• Is Homeo(\(\mathbb{RP}^2\)) strongly bounded? If not, is there a natural length function on this group?

Is it strongly bounded as a topological group (all continuous length functions bounded?)

• Schreier’s property for other groups?

Major motivating problem:

• Characterize length functions on Diff(\(M\)) up to QI. Is there a maximal word metric? (known for Homeo(\(M\)) by [10]). Properties of / examples of sequences \(\{f_n\} \subset\) Diff(\(M\)) with linear (or subexponential, etc...) growth w.r.t. all finite generating sets?
Some references


