Strong boundedness and distortion in transformation groups

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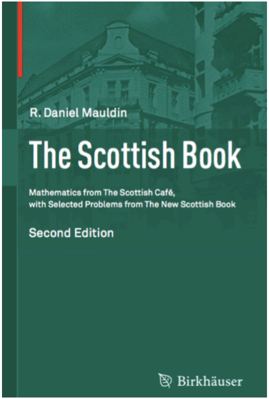
Joint work with F. Le Roux (Inst. Math. Jussieu)

Theorem (Higman–Neumann-Neumann) Γ countable group. There is a finitely generated group H (generated by two elements) with $\Gamma \subset H$.

Higman embedding theorem: Γ countable group.
There is a finitely generated group H
with $\Gamma \subset H$.

Relative version: Fix an (uncountable) group G.
(e.g. Lie group, automorphism group, homeomorphism group)
Let $\Gamma \subset G$ be countable. Is there a finitely generated $H \subset G$ with $\Gamma \subset H$?

Examples of G where the answer is always positive?



PROBLEM 111: SCHREIER

Does there exist an uncountable group with the property that every countable sequence of elements of this group is contained in a subgroup which has a finite number of generators? In particular, do the groups S_∞ and the group of all homeomorphisms of the interval have this property?

(circa 1935)

111) Problemat. Schreiera.

Czy istnieje grupa nieprzeliczalna o tej własności, że każdy ciąg przeliczalny elementów tej grupy zawarty jest w podgrupie o skończonej ilości tworzących?

W szczególności, czy własność tę posiada grupa S_∞, ~~lub~~ grupa homeomorfizmów odcinka.

Answers

Theorem (implicit in Sabbagh (1975) using Scott (1951))

Groups with this property exist.

Theorem (Galvin, 1995)

S_∞ *has this property.*

Theorem (Le Roux – M., 2016)

$\mathrm{Homeo}(I)$ *has this property.*
So does $\mathrm{Homeo}_0(M)$ *and* $\mathrm{Diff}_0(M)$ *for any manifold* M.

...and in many cases, this is a consequence of something stronger

Related concepts

Serre's (FA):

i) G doesn't split,   ii) doesn't have a $\mathbb{Z}$ quotient, and

iii) G finitely generated.

Related concepts

Serre's (FA):
i) G doesn't split, ii) doesn't have a $\mathbb{Z}$ quotient, and
iii) G not a countable increasing union of proper subgroups
implied by Schreier's property

(proof of contrapositive)

Suppose $G = \bigcup G_n$, with $G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \dots$
choose $g_i \in G_{i+1} \setminus G_i$.
$\{g_n\}$ not in any f.g. subgroup. $\square$

First examples by Koppelberg–Tits '74.
Many recent examples, starting with Bergman '04.

Related concepts

Definition

G has *strong boundedness* if every length function $G \to [0, \infty)$ is bounded. $\quad \ell(g^{-1}) = \ell(g),\ \ell(id) = 0,\ \ell(gh) \leq \ell(g) + \ell(h)$

$\Leftrightarrow$ any isometric action of G on a metric space has bounded orbits.

Definition

G has *strong distortion* if $\exists M$ and sequence $w_n \to \infty$ such that, for any $\{g_n\} \subset G$, have M-element set $S \subset G$ generating this sequence, and $|g_n|_S < w_n$.

$\Rightarrow$ subgroup distortion

$$\text{Strong distortion} \begin{array}{l} \Rightarrow \text{strong boundedness} \\ \Rightarrow \text{Schreier's property} \end{array} \not\Downarrow \quad \text{(Cornulier)}$$

For examples see [1], [4]

Distortion in transformation groups

Distortion of $G \subset \mathrm{Diff}(M) \leftrightarrow$ dynamics of action of G.

Distortion in transformation groups

- fixed points (Franks–Handel 2006):
$f \in \mathrm{Diff}(\Sigma)$, preserves μ, and distorted in some f.g. subgroup.
Then $\mathrm{supp}(\mu) \subset \mathrm{fix}(f)$.

- growth of derivatives (Calegari–Freedman)
$f(x) = x$, $\|Df_x\| > 0$ i.e. has eigenvalue of norm $\neq 1$
$\Rightarrow \langle f \rangle$ not distorted in any f.g. subgroup. of $\mathrm{Diff}(M)$, *M compact*

$\ell(f) := \|Df\|$ is unbounded length function on $\mathrm{Diff}(M)$

- "stretch" $$\ell(f) := \sup_{x,y \in \tilde{M}} |d(\tilde{f}(x), \tilde{f}(y)) - d(x,y)|$$

unbounded on $\mathrm{Homeo}(M)$ if $\pi_1(M)$ infinite.

- Much more... Polterovich [11], Hurtado [8], Gromov [5], etc.

Our results

Theorem (Le Roux – M.)

$\mathrm{Diff}^r(\mathbb{R}^n)$ *has strong distortion.* $r \neq n+1$
Given sequence g_n*, can build* S *with* $|S| = 17$ *and*
$|g_n|_S \leq 50n + 24$. *surely non optimal!*

Open question: Minimum $|S|$? Obstruction to $|S| = 2$?

Theorem false for:

- $r \geq 1$, M compact,
- $r = 0$, $\pi_1(M)$ infinite

but...

Our results

Theorem (Le Roux – M.)

M compact, or homeomorphic to interior of compact manifold. Given $g_n \subset \mathrm{Diff}^r(M)$, can build finite S that generates all g_n *(no control on word length)*.

Consequences:

1. Can do G.G.T. for countable subgroups (by including in f.g. subgroup).

Remark: G.G.T. / large-scale geometric concepts make sense in $\mathrm{Homeo}_0(M)$ by [Mann-Rosendal 15]

2. These groups are all topologically f.g., have countable dense subgroup.

3. “natural” examples of groups with Schreier's property

Proving strong distortion

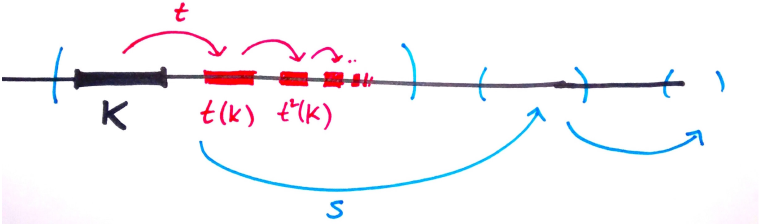
Easy case: $G = \mathrm{Diff}^0(\mathbb{R})$. ($\cong$ Homeo(I), Schrier's question)

Classical trick (G. Fisher '60)

$\{f_n\}$ supported in K compact $\subset \mathbb{R} \Rightarrow$ generated by s, t, F.

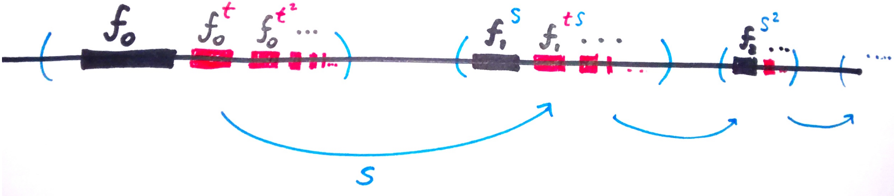
Proof:

Define t and s



Now define F consisting of (conjugate) copies of f_n

$$F = \prod_{j,n\geq 0} s^n t^j \, f_n \, t^{-j} s^{-n}$$



Check: $F \circ (tF^{-1}t^{-1}) = f_0$.

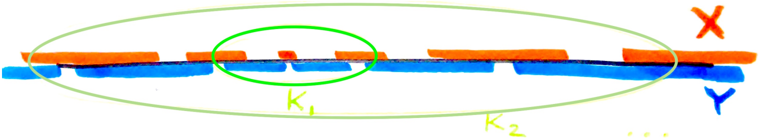
Make f_n by conjugating by power of s first: $f_n = [F^{s^{-n}}, t]$

Proving strong distortion

Lemma
Given $\{f_n\}$ sequence in $\mathrm{Homeo}_+(\mathbb{R})$. Can find X, $Y \subset \mathbb{R}$ (unions of disjoint intervals) such that

$$f_n = g_n h_n k_n$$

$\mathrm{supp}(g_n) \subset X$, $\mathrm{supp}(h_n) \subset Y$ *and* $\mathrm{supp}(k_n)$ *compact.*



idea of proof: $I \cup f(I) \subset J \Rightarrow$ can find g agreeing with f on I and identity outside neighborhood of J.

Proving strong distortion

Lemma

Given $\{f_n\}$ *sequence. Can find* X, $Y \subset \mathbb{R}$ *(unions of disjoint intervals) such that*

$$f_n = g_n h_n k_n$$

$\mathrm{supp}(g_n) \subset X$, $\mathrm{supp}(h_n) \subset Y$ *and* $\mathrm{supp}(k_n)$ *compact.*

Now use classical trick three times:

1. Find d so that $d^n k_n d^{-n}$ supported in $[0, 1]$. $\rightsquigarrow$ classical trick.
$|d^n k_n d^{-n}|_S \leq 4n + 4$.

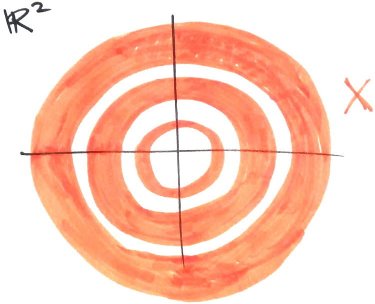
2. Trick works for X rather than K, so $g_n = [G^{s^{-n}}, t]$ for some s, t, G, $\Rightarrow g_n$ has length $4n + 4$ in finite set.

3. Same for Y, h_n.

$\square$ distortion for $\mathrm{Homeo}_0(\mathbb{R})$

Other (less easy) cases

- $\mathrm{Homeo}_0(\mathbb{R}^n)$, $\mathrm{Diff}_0(\mathbb{R}^n)$ can also do $f_n = g_n h_n k_n$



...but need to use annulus theorem in higher dimensions.

- More difficulty for $\mathrm{Diff}_0(\mathbb{R}^n)$
(classical trick doesn't produce diffeomorphisms)

$\mathrm{Diff}_0^r(\mathbb{R}^n)$ and $\mathrm{Diff}_0^r(M)$ case

Tools:

• Simplicity of $\mathrm{Diff}_c^r(M)$, $r \neq \dim(M) + 1$.

• Fragmentation: f close to identity $\Rightarrow$ product of diffeomorphisms supported on elements of open cover

Related: *Fragmentation norm*

• Replacement of classical trick using idea inspired by Avila
+ Construction of Burago–Ivanov–Polterovich
+ technical work

Open questions:

• Is Homeo($\mathbb{R}P^2$) strongly bounded? If not, is there a natural length function on this group?

Is it strongly bounded as a *topological group* (all *continuous* length functions bounded?)

• Schreier's property for other groups?

Major motivating problem:

• Characterize length functions on Diff(M) up to QI. Is there a maximal word metric? (known for Homeo(M) by [10]). Properties of / examples of sequences $\{f_n\} \subset$ Diff(M) with linear (or subexponential, etc...) growth w.r.t. all finite generating sets?

Some references

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[5] G. D'Ambra, M. Gromov, *Lectures on transformations groups: geometry and dynamics*, in Surveys in Differential Geometry, supplement to J. Diff. Geom., 1, (1991) 19-112.

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