Groups acting on the circle

rigidity, flexibility, and moduli spaces of actions

Kathryn Mann
UC Berkeley / MSRI

Γ = finitely generated group

$\mathrm{Homeo}_+(S^1)$ = group of orientation preserving homeomorphisms of S^1

$\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))$ = space of representations $\Gamma \to \mathrm{Homeo}_+(S^1)$

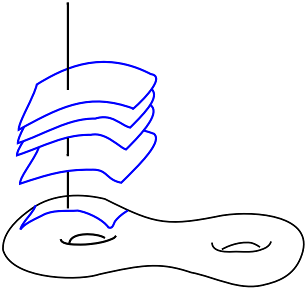
Important interpretation

$\Gamma = \pi_1(M)$, e.g. $M =$ 

$\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1)) =$ space of *flat* S^1 bundles over M

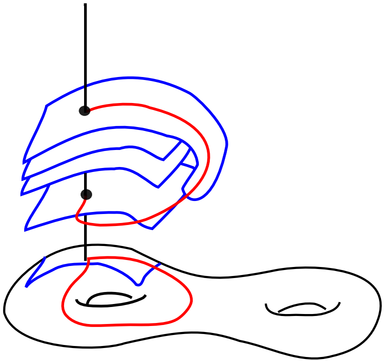
has *flat connection*

foliation transverse to fibers



Important interpretation

flat bundle $\underset{monodromy}{\longleftrightarrow}$ $\rho : \Gamma = \pi_1(M) \to \mathrm{Homeo}_+(S^1)$



$\gamma \in \Gamma$ $\quad \rho(\gamma) \circlearrowleft S^1$ = fiber over basepoint

flat bundles/equivalence

$\leftrightarrow \mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/$(semi-)conjugacy

Basic Problem

Understand $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$

1. Nontrivial?
   - Does Γ act (nontrivially) on S^1?
     (faithfully)
   - (more refined) *Does a S^1 bundle admit a flat connection?*

2. Describe
   - connected components
     $\leftrightarrow$ deformation classes of $\left\{ \begin{array}{l} \text{flat bundles} \\ \text{actions} \\ \text{representations} \end{array} \right.$
   - isolated points
     $\leftrightarrow$ rigid representations

3. Parameterize $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$
   ? give "coordinates" local coordinates?

Examples to keep in mind

- $\Gamma = \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$
- $\Gamma \to S^1 \subset \mathrm{Homeo}_+(S^1)$
- Free group ... generators act by arbitrary homeomorphisms
- More sophisticated examples
  e.g. $\pi_1(M^3)$ – if M^3 has pseudo-Anosov flow, can build faithful $\pi_1(M^3) \to \mathrm{Homeo}_+(S^1)$

Coordinates on $\mathrm{Hom}(\Gamma, \cancel{\mathrm{Homeo}_+(S^1)})/\sim$

Too hard!
For motivation, look instead at easier space

$$\mathrm{Hom}(\Gamma,\ \mathrm{SL}(2,\mathbb{R}))/\sim$$

Trace Coordinates on $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{R}))/\sim$

obvious facts:

$\mathrm{tr} : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$

- conjugation invariant $\mathrm{tr}(ghg^{-1}) = \mathrm{tr}(h)$
- not a homomorphism

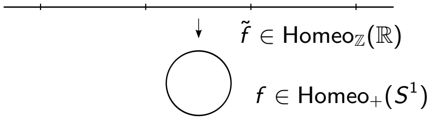
Theorem: $\rho_1,\ \rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{R}))$ *nondegenerate*.

If $\mathrm{tr}(\rho_1(\gamma)) = \mathrm{tr}(\rho_2(\gamma))\ \forall\gamma \in$ a finite set, then $\rho_1 \sim \rho_2$.

Coordinates on $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$

? Conjugation-invariant function $\mathrm{Homeo}_+(S^1) \to \mathbb{R}$

Definition (Poincaré)



translation number $\tau(\tilde{f}) := \lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n}$

- Conjugation invariant (Poincaré)
- not a homomorphism (exercise)

OOPS! depends on lift $\tilde{f}$

Coordinates on $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$

Two solutions

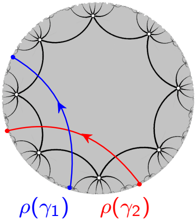
i) $\tau \bmod \mathbb{Z}$ does not depend on lift. $\lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n} \bmod \mathbb{Z}$

$\tau_{mod\mathbb{Z}}$: $\mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$

ii) define $c(f,g) := \tau(\tilde{f}\tilde{g}) - \tau(\tilde{f}) - \tau(\tilde{g})$
does not depend on lifts.

$\tau_{mod\mathbb{Z}}$ does not give coordinates.

e.g. $\Gamma = \pi_1(\Sigma_g)$, Fuchsian rep $(\mathrm{PSL}(2,\mathbb{R}))$



$\tau_{mod\mathbb{Z}}(\rho(\gamma)) = 0$ for all γ!

A cocycle

$c(f, g)$ satisfies cocycle condition. $[c] \in H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{R})$

the Euler class

Given $\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$, $\quad \rho^*[c] \in H_b^2(\Gamma; \mathbb{R})$

Theorem (Ghys, Matsumoto)

$\rho \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))/\sim$ *is determined by* $\rho^*[c] \in H_b^2(\Gamma, \mathbb{R})$ *and value of* $\tau_{mod\mathbb{Z}}(\rho(\gamma))$ *on generators for* Γ*.*

Applications

- Milnor–Wood [Wo]: Existence

  $\begin{array}{ccc} S^1 & \to & E \\ & & \downarrow \\ & & \Sigma \end{array}$ admits a flat connection $\Leftrightarrow$ $|\text{Euler number}| \leq |\chi(\Sigma)|$

  characteristic class of bundle

- Matsumoto [Mat87]: Rigidity

  $\rho : \pi_1(\Sigma) \to \text{Homeo}_+(S^1)$ has maximal Euler number

  $\Leftrightarrow$ semi-conjugate to Fuchsian

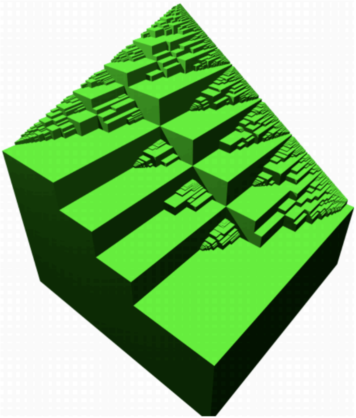
- Calegari [Ca]: Rigidity

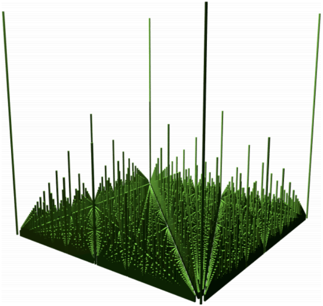
  examples of other groups Γ with few/rigid actions on S^1

- Calegari–Walker [CW]: Pictures

  slices of $\text{Hom}(F, \text{Homeo}_+(S^1))$ in "transation number coordinates"

  F = free group





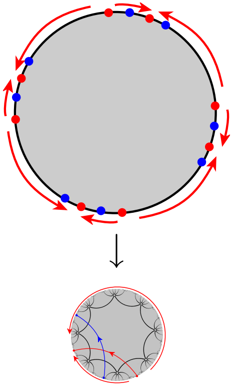
Calegari, Walker *Ziggurats and rotation numbers* [CW]

Applications, cont.

- Mann [Man14]: Connected components, Rigidity

  · New examples of rigid representations $\pi_1(\Sigma_g) \to \text{Homeo}_+(S^1)$.

  (non-maximal Euler number)



lift to k-fold cover of S^1

· Identification/classification of more connected components of $\text{Hom}(\pi_1(\Sigma_g), \text{Homeo}_+(S^1))/\sim$

Open questions

1. Does $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(S^1))/\sim$ have infinitely many connected components?

   $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))/\sim$ has finitely many, classified by Goldman [Go]

2. Are there more examples of rigid representations in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(S^1))/\sim$?

3. *Is the space of foliated $S^1 \times \Sigma$ products connected*?

   (flat bundles with Euler number 0)

4. Is $\mathrm{Hom}(\pi_1(\Sigma)), \mathrm{Homeo}_+(S^1))$ locally connected?

5. Groups other than $\pi_1(\Sigma)$?

Another perspective

Question: Does Γ act nontrivially/faithfully on S^1?

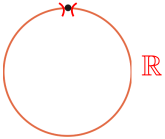
Theorem: Γ acts faithfully on $\mathbb{R} \Leftrightarrow \Gamma$ is Left-orderable

$\exists$ total order $<$ on Γ; $a < b \Leftrightarrow ga < gb$

Example: $\mathbb{R}$

Application: (Witte Morris [Mo])

$\Gamma < \mathrm{SL}(n, \mathbb{Z})$ *finite index* $_{n \geq 3} \Rightarrow \Gamma$ *has no faithful action on* S^1

idea: 

Open: $\Gamma < \mathrm{SL}(n, \mathbb{R})$ lattice $_{n \geq 3}$. Has faithful action on S^1?
Are all actions *finite*?

Many partial/related results known (see references in [Mo])

Another perspective

Does Γ act nontrivially on S^1?

Theorem: Γ acts faithfully on $\mathbb{R} \Leftrightarrow \Gamma$ is Left-orderable

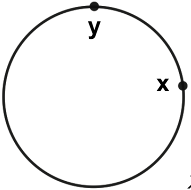
$\exists$ total order $<$ on Γ; $a < b \Leftrightarrow ga < gb$

Example: $\mathbb{R}$

Theorem: Γ acts faithfully on $S^1 \Leftrightarrow \Gamma$ is circularly-orderable

... ?

Example: S^1



$x < y$??

Another perspective

Does Γ act nontrivially on S^1?

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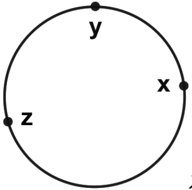
... ?

Example: S^1



$x < y < z$

(x, y, z) is positively oriented

orientation of triples is left-multiplication invariant

Circular orders

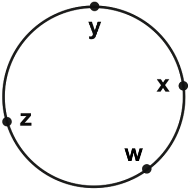
Definition

A circular order on Γ is a function $ord : \Gamma \times \Gamma \times \Gamma \to \{\pm 1, 0\}$

$(x, x, y) \mapsto 0$

$(x, y, z) \mapsto \pm 1$ (orientation)

satisfying a compatibility condition on 4-tuples



(exercise!)

A new perspective on the old perspective

"compatibility condition" on 4-tuples is the cocycle condition !

$[ord] \in H_b^2(\Gamma; \mathbb{Z})$

(recall)

Theorem: Γ has circular-order

$\Leftrightarrow \exists$ faithful $\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$

Theorem: (Thurston, Ghys, ...) $[ord] = 2\,\rho^*[c]$ in $H_b^2(\Gamma; \mathbb{R})$

Homework

Describe the actions of Γ on S^1
$\circlearrowleft$ [your favorite group]

interesting "geometric" examples:

- $\Gamma =$ lattice in semi-simple Lie group
- $\Gamma = \pi_1(M^3)$
  foliations, anosov flows, universal circles...
- $\Gamma = \mathrm{MCG}(\Sigma_{g,*})$
- $\Gamma = \mathrm{MCG}(\Sigma_{g,b})$
- $\Gamma = \pi_1(\Sigma_g)$
- etc...

see [CD] (3-manifold case), [Mo] (lattices), and [Th] (3-manifolds, etc.) for a start...

Epilogue

What I didn't say: Other perspectives on group actions on the circle

- Semi-conjugacy versus conjugacy. (nice intro in [BFH]) (also relates to regularity issues, see below)
- Regularity: Compare $\mathrm{Hom}(\Gamma, G)$ where $G = \mathrm{Diff}^r(S^1)$ or $G = \mathrm{Homeo}(S^1)$ or $G = \mathrm{PSL}(2, \mathbb{R})$. What about $G = QS(S^1)$?...
  (Goldman [Go] for $\mathrm{PSL}(2, \mathbb{R})$, Bowden [Bo] and Navas [Na] for Diff^r, Ghys ...)
- Many other perspectives on bounded cohomology, e.g. *continuous bounded cohomology*, and applications to actions on S^1
  ([Bu] and references there)
- Tools from *low dimensional dynamics*, often applicable in higher regularity case. In Homeo case, new ideas in [Mat14] may be promising.
- This talk focused on Γ... but can we understand $\mathrm{Homeo}_+(S^1)$ better as a group? How to think of it as an "infinite dimensional Lie group"? What about $\mathrm{Diff}_+(S^1)$ (truly a ∞-dimensional Lie group)?
  What is the *algebraic structure* of these groups, and how does it relate to their *topological structure?* (see e.g. [Man15])

Some references and recommended reading

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