Comments on HW 2

GP 1.1 #4.

Comments: Many students wrote down an inverse function, but did not justify why it was an inverse. You need to check, e.g. show that $f \circ f^{-1}(x) = x$.

The best way to show that some of these functions are smooth is to say they are compositions, products, sums, etc. of smooth functions. For instance, you already know that polynomials are smooth, and $\sqrt{x}$ is smooth away from 0, etc. It's okay to use basic calculus facts in this course.

GP 1.1 #6.

Hint: check the derivative of the inverse at zero!

GP 1.1 #10.

This problem gave many students some trouble! Here is an outline of a good proof.

1. Define a map $S^1 \times S^1 \to X$. There are many ways to do this, one way is $(x, y, z, w) \mapsto ((a + bz)x, (a + bz)y, bw)$. This is a map from $S^1 \times S^1 \subset \mathbb{R}^4$ to $\mathbb{R}^3$ whose image is X (check!)
2. Verify that X is smooth, by *defining* an extension to an open set containing $S^1 \times S^1$ (which might have the same formula as the map that you gave) and stating why it is smooth.
3. Write down a formula for an inverse for your map, and verify that it is smooth, in a similar way as above.

GP 1.1 #18

See the "information on bump functions" on the course website for an extremely detailed solution (I was not expecting you to have something this detailed).

Problem 2: if this was conceptually difficult, to convince yourself that f is smooth, you might want to draw the graph of the x-coordinate and y-coordinate functions.

GP 1.2 # 5. Notes: There *is* a bijection $\mathbb{R}^k \to \mathbb{R}^l$ for any k and l. So saying that there is no bijection is not enough. It is true that $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic when $k \neq l$, but that is a very hard theorem from topology (it is called "invariance of domain", look it up!). Here is a proof outline for you:

Suppose $f : \mathbb{R}^k \to \mathbb{R}^l$ was a diffeomorphism. Then df_0 would be a linear isomorphism between tangent spaces. The tangent space to $\mathbb{R}^k$ at 0 is $\mathbb{R}^k$, and the tangent space to $\mathbb{R}^l$ at 0 is $\mathbb{R}^l$. These are only isomorphic as vector spaces if $k = l$.

GP 1.2 #12. This was generally well done, although it appears that a couple of you might have looked for a solution online and copied it... I'll be checking for this carefully in the future.