Day 1: Wrinkly paper

Exercise 1 (Geodesics on the sphere – Optional)**.** Show that the geodesics on a sphere are exactly the great circles by looking at straight lines on better and better polyhedral approximations of the sphere. (The approximations in the figure below are given by successively subdividing each triangular face into 4 equilateral triangles.)

Exercise 2 (Geodesics on $\mathbb{H}^2$)**.** Build a paper model of $\mathbb{H}^2$ with 7 equilateral triangles around a vertex. Draw some geodesics on your paper model of $\mathbb{H}^2$. What do they look like? More specifically:

1. Draw two geodesics which start out as parallel lines, but eventually diverge
2. Draw a geodesic that contains a segment parallel to, and very close to, the edge of one of the equilateral triangle pieces. Notice how it follows a path made up of edges. We'll call these special geodesics *edge-path geodesics*.
3. Describe all possible edge-path geodesics. What paths of edges can they follow?
4. Draw a big triangle on $\mathbb{H}^2$ with all sides geodesics.
5. Can you draw a (big) geodesic sided rectangle? (i.e. four sides, 90 degree corners). Drawing it inside a single flat triangle piece or two... or 3... is cheating! If you can't, can you *prove* that it's impossible to do so? Hint/further question: What is the angle sum of your big geodesic triangle?

Exercise 3. (Area and circumference in $\mathbb{E}^2$ and $\mathbb{H}^2$)

1. Find a formula for the combinatorial area and circumference of a hexagonal "disc" of radius r in $\mathbb{E}^2$.
2. Find a formula for the combinatorial circumference of a triangulated disc of radius r in $\mathbb{H}^2$
3. Find a (recursively defined or explicit) formula for the combinatorial area of a triangulated disc of radius r in $\mathbb{H}^2$. It may help to draw a flat picture of a space with 7 triangles around a vertex (a *regular degree 7 graph*).

Exercise 4. (challenge) Assume our equilateral triangles have side length 1. Which points on the circumference of the triangulated disc of radius r in $\mathbb{H}^2$ are distance (very close to) r from the center along edge-path geodesics? Why does this make our triangulated disc a good approximation of the genuine disc of radius r in $\mathbb{H}^2$?

Day 2: Mapping the hyperbolic plane

Exercise 5. Describe the image of great circles under stereographic projection.

Exercise 6. (Geodesics in the disc)

1. Draw the image of some geodesic segments on the Poincaré disc by copying them off of your paper model.
2. Can you characterize what all edge-path geodesics look like?
3. What does this tell you about straight lines in the Poincaré disc model? Does this remind you of any of the projections of the sphere?

Exercise 7. Using the fact that Möbius transformations map circles to circles *and* preserve angles, describe the geodesics of $\mathbb{H}^2$ in the upper-half plane model.

Exercise 8 (optional, challenge)**.** Look at Jos Leys's pictures at http://www.josleys.com/show_gallery.php?galid=325. Can you figure out what Leys' method is (not the technical details, just the general strategy) for turning a tiling of the Euclidean plane into one on the hyperbolic plane? (Later, we'll make some tessellations of $\mathbb{H}^2$ using different methods, but feel free to start exploring now)

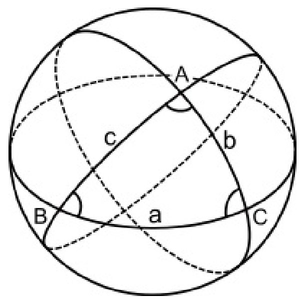
Exercise 9 (optional, harder)**.** Prove that

1. Cylindrical projection preserves area, and/or that
2. Stereographic projection preserves angles and sends circles to circles.

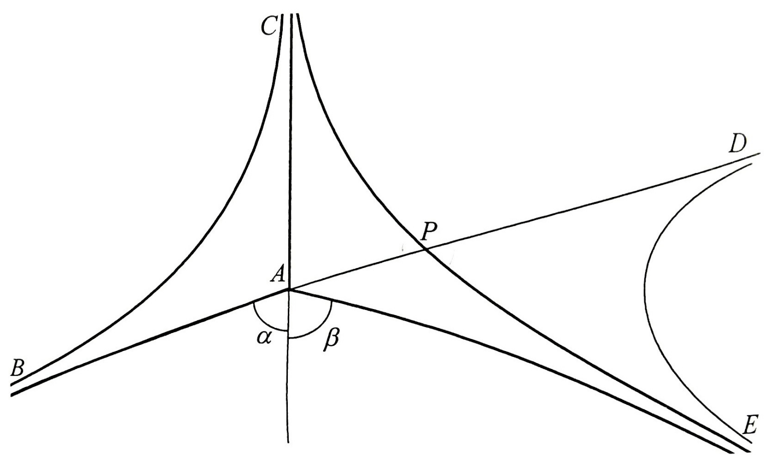
Day 3: Hyperbolic and spherical triangles, *curvature*

Exercise 10. (Spherical triangles)

1. What is a formula for a triangle with angles α, β, γ on a sphere of radius R?
2. What angle sums are possible?



Exercise 11. Let $A(\alpha)$ denote the area of an One-Angle-Triangle in $\mathbb{H}^2$ of exterior angle α. Show that A is additive, i.e. that $A(\alpha) + A(\beta) = A(\alpha + \beta)$. Hint below!



Exercise 12. Using the fact that $A(\alpha) = \lambda\alpha$, derive a formula for a hyperbolic triangle with angles a, b, c in terms of λ. Note that in particular, you just proved that the angles of a triangle determine the area!

Exercise 13. Derive a formula for the area of a polygonal region with geodesic sides (on the unit sphere, and in $\mathbb{H}^2$) in terms of the angles at the vertices, by dividing the region up into triangles.

Day 4: Tilings, isometries, and hyperbolic soccer

Exercise 14. Build a hyperbolic soccer ~~ball~~ plane. You can find detailed instructions and a template here: http://theiff.org/images/IFF_HypSoccerBall.pdf

Exercise 15. (Challenge: What is the shape of a hyperbolic soccer ball?) Is it possible to make a real *ball* with heptagons and hexagons in the hyperbolic soccer pattern? Is it possible to make any closed shape? If so, what would it look like? Start by trying to build a "cylinder" and experiment from there...

Exercise 16. Show that the dual of the dual of a polyhedron (or tiling) is the original polyhedron (tiling).

Exercise 17. Prove that every symmetry of a polyhedron is also a symmetry of its dual polyhedron. Prove that every symmetry of a tiling of the plane is also a symmetry of its dual. Show the same is also true for the hyperbolic plane. (this should help you understand isometries of $\mathbb{H}^2$)

Exercise 18. (optional, assumes you know some group theory.) Describe the *groups of symmetries* of each of the Platonic solids. Which have the same symmetry group? Which are duals of each other?

Exercise 19. What $\pi/p, \pi/q, \pi/r$ triangles exist in $\mathbb{E}^2$? What tilings do they give? What about in $\mathbb{H}^2$? Can you visualize what these tilings look like?

Exercise 20 (Tiling with regular polygons)**.** . Define a *regular polygon* to be a polygon with all sides equal length, and all angles equal.

1. Show that you can tile the hyperbolic plane with regular 4-gons (squares), with 5 around each vertex.
2. Show that you can tile the hyperbolic plane with regular 4-gons, with n around each vertex, for any $n \geq 5$
3. Show that you can tile the plane with regular n-gons. How many must you put around each vertex (you have options). In each case, what must the interior angle of the n-gon be?

Exercise 21 (Symmetry groups, optional)**.** If you took Don's *reflection groups* class (or paid very close attention to Frank Farris's *polyhedral symmetry* talk), you know about reflection groups in $\mathbb{E}^2$. What can you say about reflection groups in $\mathbb{H}^2$? Can you get *new*, *different* groups? Start by trying to understand the group created by reflections in the sides of the $\pi/2, \pi/3, \pi/7$ triangle.

Exercise 22. Create some original M.C. Escher-like tilings of hyperbolic space!