30-minute Test 1

Math 141
Spring 2016

Name.. Student ID number ....

Instructions: This test has 4 questions for a total of 16 points. Please show your work in the space provided.

Question	Points	Score
1	3	
2	2	
3	5	
4	6	
Total:	16	

1. (3 points) Let X and Y be manifolds. A function $f : X \to Y$ is called a *local diffeomorphism* if...

   for each point $x \in X$, there is a neighborhood U of x in X, and a neighborhood V of $f(x)$ in Y such that the restriction of f to U is a diffeomorphism onto V.

   alternatively, you can say "... an open set U in $\mathbb{R}^n$ containing x, and an open set V in $\mathbb{R}^m$ containing $f(x)$, such that the restriction of f to $U \cap X$ is a diffeomorphism onto $V \cap Y$".

2. (2 points) Let X and Y be manifolds, and let $f : X \to Y$ be a smooth map.

   a) Let $x \in X$. The derivative df_x is a function from $T_x(X)$ to $T_{f(x)}(Y)$

   b) f is called a *submersion* at x if df_x is surjective

3. (a) (3 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = xyz$. Show that $f^{-1}(1)$ is a submanifold of $\mathbb{R}^3$.

   By the theorem about pre-images, we just need to show that 1 is a regular value of f, i.e. that df_p is surjective at every point p with $f(p) = 1$. Let $p = (x, y, z) \in f^{-1}(1)$. Then $xyz = 1$, so in particular x and y and z are all nonzero. Now $df_p = (yz, xz, xy)$ which is surjective since yz, xz and xy are all nonzero. Thus, 1 is a regular value.

   (b) (2 points) Describe the linear subspace of $\mathbb{R}^3$ that is the tangent space to $f^{-1}(1)$ at the point $(1, 1, 1)$.
   The tangent space is the kernel of the derivative df_p at $p = (1, 1, 1,)$.
   Since $df_p = (1, 1, 1)$, its kernel is spanned by the vectors $(1, -1, 0)$ and $(1, 0 - 1)$.

4. Let $f : S^1 \to S^1$ be an immersion.

   (a) (4 points) Prove that $f(S^1)$ is an open set in S^1.

   Since f is an immersion, its derivative at every point is injective. Since both the source and target manifolds have the same dimension, this means that df_x is an isomorphism, so f is actually a local diffeomorphism. It follows from the definition of local diffeomorphism that $f(S^1)$ is an open set in S^1 – at any point $f(x)$ in the image, there is an open neighborhood of that point in the image (the neighborhood V from the answer to the first question).

   (b) (2 points) Show that f is surjective. (possible approach: use some techniques from topology)

   Since S^1 is path-connected and compact and f is continuous, $f(S^1)$ must be path connected and compact. So it is either S^1, a point, or a closed interval in S^1. Since the image is open by part (a), it must be S^1.

5. Bonus (0.5 points). Is f (from the question above) necessarily injective? Explain.

   No, consider f defined by $f(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta))$.