Math 141 Practice Exam

Notes: This practice exam is a version of the Fall 2014 Math 141 final exam, modified to match the material that we covered in our class. It should be roughly comparable in length and difficulty to our final, although I will break more problems into steps, i.e. as "part a), part b), part c)" and not have so many true/false questions. (although these ones are good practice!) Your midterm will give you a better idea of the format of my exams.

1. Define the following terms.
   (a) (3 points) immersion

   (b) (3 points) homotopic maps

   (c) (3 points) $T_x(X)$, when X is a manifold with boundary and $x \in \partial X$

   (d) (3 points) mod 2 winding number

2. Determine whether the following statements are true or false. No justification is required.

(a) (3 points) Every k-dimensional manifold has an embedding into $\mathbb{R}^{2k+1}$.

TRUE **FALSE**

(b) (3 points) If $(X, \mathcal{T})$ is a connected topological space, then for any $x, y \in X$, there exists a continuous path from x to y.

TRUE **FALSE**

(c) (3 points) Homotopy is an equivalence relation.

TRUE **FALSE**

(d) (3 points) If $f : S^k \to \mathbb{R}^{k+1}$ is a smooth map whose image does not contain the origin and $f(-x) = -f(x)$ for all $x \in S^k$, then $W_2(f, 0) = 1$.

TRUE **FALSE**

(e) (3 points) Every compact, connected smooth 1-manifold is diffeomorphic to S^1.

TRUE **FALSE**

(f) (3 points) If f is a smooth map from the open unit ball $B^n \subset \mathbb{R}^n$ into itself, then there exists a point $x \in B^n$ such that $f(x) = x$.

TRUE **FALSE**

(g) (3 points) For any smooth map $f : X \to Y$, where X and Y have the same dimension, we have $\deg_2(f) = \deg(f) \bmod 2$.

TRUE **FALSE**

3. Compute the following. Explain your computations, but a rigorous proof is not required.

   (a) (4 points) $\deg_2(f)$, $f : S^1 \to S^1$ is the identity

   (b) (4 points) $\deg_2(f)$, $f : S^1 \to \mathbb{R}$ is the projection onto the x-axis

   (c) (4 points) $I_2(X, Z)$, where X and Z are circles of unit radius in $\mathbb{R}^2$, X is centered at $(-1, 0)$, and Z is centered at $(1, 0)$.

4. (a) (8 points) Let $f : H^3 \to \mathbb{R}$ be given by $x^2 + y^2 + xz$. Show that $S = f^{-1}(1)$ is a manifold with boundary, and determine its boundary.

(b) (6 points) Let f be as in the previous question. Let $\mathbb{R}$ have the standard orientation, and H^3 the standard orientation as a subset of $\mathbb{R}^3$. Then S can be given a pre-image orientation. Write explicitly a positive basis for $T_x(S)$, where $x = (0, 1, 0)$.

For your reference: If $f : X \to Y$, $Z \subset Y$, $f(x) = y$ and $f^{-1}(Z) = S$, we define pre-image orientation using the conventions

$$N_x(S, X) \oplus T_x(S) = T_x(Y)$$

$$df(N_x(S, X)) \oplus T_y(Z) = T_y(Y)$$

5. (10 points) Show that $\mathbb{R}^2$ and the cylinder $S^1 \times \mathbb{R}$ are not diffeomorphic.

6. (a) (10 points) Let $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ denote the non-zero real numbers. Show that the function $F : \mathbb{R}_* \times \mathbb{R}^3 \to \mathbb{R}^3$, given by $F(t, v) = p + tv$ where $p \in \mathbb{R}^3$ is fixed, is a submersion.

(b) (10 points) Fix $p \in \mathbb{R}^3 \setminus S^2$. Show that almost every line through p intersects $S^2 \subset \mathbb{R}^3$ transversally.

7. (5 points) Let X be a smooth manifold with boundary, and $x \in \partial X$. Show that there exists a smooth non-negative function $f : U \to \mathbb{R}$ on an open subset $U \subset X$ containing x such that $f(z) = 0$ if and only if $z \in U \cap \partial X$. (Hint: Consider $X = H^k$.)