Midterm Exam

Math 141

Spring 2016

Name Solutions / Comments Student ID number ....

Instructions: This test has 7 questions for a total of 40 ~~points. Please show your work~~ in the space provided. There is an extra page at the back for scratch work or extra space.

Question	Points	Score
1	4	
2	4	
3	3	
4	3	
5	9	
6	9	
7	8	
Total:	40	

1. (4 points) Which of the following sets have measure zero? Circle all that apply:

   a) The rational numbers in $\mathbb{R}$

   b) Any bounded subset of $\mathbb{R}^k$

   c) The set of critical values of any smooth function — Sard's theorem

   d) The set $f^{-1}(z)$ in X, where f is smooth, X is a manifold, and z is a regular value of $f : X \to \mathbb{R}$.

   we know that $f^{-1}(z)$ is a manifold of smaller dimension than X so has measure zero in X.

   e) The image of a local submersion.

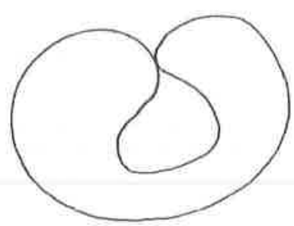
   always contains an open set, so not measure zero

2. (4 points) This picture is the image of a continuous function $f : S^1 \to \mathbb{R}^2$.

   Based on the picture, f could be ...

   ☑ smooth
   ☑ an immersion
   ☐ an embedding — not one-to-one
   ☐ a local diffeomorphism — not to $\mathbb{R}^2$ (it's a local immersion)

   (check all that apply)

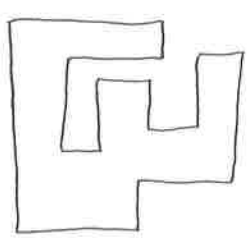


This picture is the image of another continuous function $g : S^1 \to \mathbb{R}^2$.

Based on the picture, g could be ...

☑ a homeomorphism onto its image
☑ possibly not injective (go around twice!)
☑ smooth
☐ an immersion — no corners allowed

(check all that apply)



remember, the image of a smooth map can have corners if the derivative is zero somewhere

5. (a) (3 points) Find all the critical values of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 + xz - 4z + 3$.

$df_{(x,y,z)} = (2x+z,\ 2y,\ x-4)$

if not surjective to $\mathbb{R}$, then $y=0,\ x=4,\ z=-8$.

$\cancel{f(4,0,-8) = 16+0+4(-8)+4(8)+3 = 19}$

This is the only critical value.

(b) (2 points) Let $X = \{(x, y, z) : x^2 + y^2 + xz - 4z \geq -3\} \subset \mathbb{R}^3$. Why does part a) show that X is a manifold with boundary? State any theorem you use to justify this.

Theorem (lemma?); If Y is a manifold, and 0 a regular value of $f : Y \to \mathbb{R}$, then $f^{-1}([0,\infty))$ is a manifold with boundary. (and $\partial(f^{-1}([0,\infty))) = f^{-1}(0)$)

In our case, $Y = \mathbb{R}^3$ and f is the function from part a). Since 0 is not a critical value, $f^{-1}([0,\infty)) = X$ is a manifold with boundary.

(c) (1 point) Let X be the manifold with boundary in part b). What is ∂X?

$f^{-1}(\{0\})$ or $\{(x,y,z) \mid x^2+y^2+xz-4z = -3\}$

(d) (1 point) For X as above, what is the dimension of $T_x(\partial X)$ for $x \in \partial X$? (no justification needed) 2

(e) (2 points) And what is $T_x(\partial X)$ at $x = (0, 1, 1)$?

$df_{(0,1,1)} = (1, 2, -4)$, so

$T_x(\partial X)$ is the orthogonal complement to the vector $(1,2,-4)$ in $\mathbb{R}^3$, (translated to pass through $(0,1,1)$)

OR The plane spanned by $(-2,1,0)$ and $(4,0,1)$ (translated to $(0,1,1)$)

[I didn't deduct points if you forgot to say "through $(0,1,1)$"]

3. (3 points) Which of the following examples of X and Z are transversal submanifolds of $\mathbb{R}^3$?
   Circle all that apply.

   a) X = the x-axis, and Z = the z-axis
   b) X = the unit sphere and Z = the z-axis
   c) X = the x-axis, and $Z = f(Y)$ where f is a submersion of some manifold Y.
   d) the manifolds in part c) might be, but are not always transversal, it depends on f.

Explanation: for each point $z \in f(Y)$, the local submersion theorem says there is an open set U in $\mathbb{R}^3$ containing z and contained in $f(Y)$, So $f(Y)$ is an open subset (i.e. a submanifold) of $\mathbb{R}^3$ with dimension 3,

4. (3 points) Give **ONLY ONE** of the following definitions:
   A) a property of a smooth function $f : X \to Y$ is *stable* means that ...
   OR
   B) Let $X \subset \mathbb{R}^n$ be a manifold. The tangent bundle of X is...

A) A property is stable if, for any homotopy $F : X \times I \to Y$ with $f(x) = F(x, 0)$ there exists $\varepsilon > 0$ such that, for all $t < \varepsilon$, the function $f_t : X \to Y$ given by $f_t(x) = F(x, t)$ has the property.

6. (a) (3 points) Suppose f and g are smooth functions from X to Y. What does it mean for f to be *homotopic* to g?

There exists a smooth function $F: X \times [0,1] \to Y$ such that $f(x) = F(x,0)$ for all x, and $g(x) = F(x,1)$ for all x.

(b) (6 points) Let Y be a manifold with the property that every smooth function $f: S^1 \to Y$ is homotopic to a constant function (in other words, Y is "simply connected"). Prove that $Y \times \mathbb{R}$ also has this property.

Let $f: S^1 \to Y \times \mathbb{R}$ be smooth.

Then we may write $f(x) = (g(x), h(x))$ where $g(x) \in Y$, $h(x) \in \mathbb{R}$, and both are smooth.

By assumption, there exists a homotopy $F: S^1 \times [0,1] \to Y$ from g to a constant function.

Let $G: S^1 \times [0,1] \to Y \times \mathbb{R}$ be defined by

$$G(x,t) = \left(F(x,t),\ (1-t)h(x)\right)$$

G is smooth since $F(x,t)$ is, and $h(x)$ is, and

$$G(x,0) = (F(x,0),\ h(x)) = (g(x), h(x)) = f(x)$$ for all x

$$G(x,1) = (F(x,1),\ 0)$$ a constant function.

7. (8 points) Let $f : X \to Y$ be a smooth map, and suppose that x is a point in X such that df_x is injective, but not surjective. Without using Sard's theorem, show that x has a neighborhood U so that $f(U)$ has measure zero in Y. Give a clear and precise proof.

   You may use the fact that $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$ when $k < l$, but do not use other results proved in homework.

Suppose $\dim(X) = k$, $\dim(Y) = l$

Since df_x is injective, the local immersion theorem says that there exists a neighborhood U of x in X and neighborhood V of $f(x)$ in Y and charts $\phi : U \to \mathbb{R}^k$, $\psi : V \to \mathbb{R}^l$ such that

$\psi \circ f \circ \phi^{-1} : \underset{\text{open in } \mathbb{R}^k}{\phi(U)} \to \underset{\text{open subset of } \mathbb{R}^l}{\psi(V)}$ is the canonical immersion

Since df_x is not surjective, $k < l$. (if $k = l$, the canonical immersion would be the identity, so df an isomorphism)

Since $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$,

$\psi \circ f \circ \phi^{-1}(\phi(U))$ has measure zero in $\psi(V) \subset \mathbb{R}^l$, since $\psi(V)$ is an open set.

Then $f(U)$ is the image of this measure zero set under the parametrization ψ^{-1}, so by definition of measure zero in manifolds, $f(U)$ has measure zero in Y.