Math 141 Homework 7

Reading for this week:
GP section 1.8. – you don't need to worry about all the details of Whitney embedding for non-compact manifolds, just the general idea.
GP section 2.1

Problems to hand in

1. Do the following problems from GP section 1.8. 4, 5 (in class, we called this the "footprint map"), 9 (use the hint and the regular value theorem).

2. Do the following problems from GP section 2.1:
   1, 2, 3, 9.

3. The following sequence of problems explains the idea of how to produce a proper map from a manifold to $\mathbb{R}$.
   (a) Recall the "bump function" defined in problem 18 of GP 1.1. In that problem, you showed that for any $0 < a < b$, there exists a smooth function on $\mathbb{R}^k$ that is 1 on the ball of radius a, zero outside the ball of radius b, and strictly between 0 and 1 at intermediate points.
   Let $\mu_j : \mathbb{R} \to \mathbb{R}$ be a bump function that is 0 outside of $(j-2, j+2)$ and 1 on $(j-1, j+1)$.
   Let $f : \mathbb{R} \to \mathbb{R}$ be the function
   $$f(x) = \sum_{j \in \mathbb{Z}} |j| \mu_j(x).$$
   Show that this sum is finite for every x (so that f is well-defined), that f is smooth, and that f is proper.

   (b) Now we generalize to higher dimensions. Let $p_1, p_2, p_3, \ldots$ be a list of all the points in $\mathbb{R}^k$ with integer coordinates. For example, the point $(1, 2, 3, -4) \in \mathbb{R}^4$ counts, but not $(0.37, 2, 3, 5)$. Note that there are countably many such points, so it is possible to make such a list.
   Let $\mu_j : \mathbb{R}^k \to \mathbb{R}$ be a bump function that is 0 outside of the ball of radius $k+1$ centered at p_j, and 1 on the ball of radius k centered at p_j. Show that
   $$f(x) = \sum_{j=1}^{\infty} j \mu_j(x)$$
   is well defined, smooth, proper, and never zero.

   (c) *(partitions of unity)* Using the functions μ_j from the previous part, define
   $$\theta_j(x) = \frac{\mu_j(x)}{\sum_{j=1}^{\infty} \mu_j(x)}$$
   Show that θ_j has the following properties (these should be easy)
   - $0 \leq \theta_j(x) \leq 1$ for all x
   - At each point x, all but finitely many θ_j are zero. In fact, θ_j is zero outside the ball of radius $k+1$ about p_j
   - For each x, $\sum_{j=1}^{\infty} \theta_j(x) = 1$

Remark: A collection of functions θ_j on a manifold X with these properties is called a "partition of unity". On page 52 of GP, it is proved that such a collection exists for any manifold, and instead of "ball of radius 2 about p_j" one can choose any collection of open sets $B_1, B_2, ...$ that cover X and have θ_j be zero outside of B_j

(d) **(bonus question, not to hand in)** Now let X be an arbitrary manifold. Generalize our strategy for $\mathbb{R}^k$ to produce a proper map from X to $\mathbb{R}$. The solution proposed in GP uses partitions of unity.