Math 141 Homework 10

Reading for this week:
GP section 2.4.

Problems to hand in

1. (a) At the end of GP section 2.3 (page 72), there is a lemma about functions that are zero on closed sets. Prove the following related lemma:
   Let $C \subset \mathbb{R}^n$ be a closed set. Prove that there is a continuous function $f : \mathbb{R}^n \to [0, \infty)$ such that $f(x) = 0$ if and only if $x \in C$.
   (hint: show that the function $f(x) = \inf\{\|x - c\| : c \in C\}$ is continuous by using an $\epsilon - \delta$ argument).

   (b) Now suppose $X \subset \mathbb{R}^n$ is a manifold, and $C \subset X$ is a closed set. Show that you can find $f : \mathbb{R}^n \to [0, \infty)$ such that $f(x) = 0$ if and only if $x \in C$. (hint: restrict the function from part a.)

   (c) (Not to hand in) By using a partition of unity (see HW 7), you can in fact prove that there is a *smooth* function that is zero exactly on C, whenever C is a closed set. For a challenge, give an *explicit description* (i.e. don't say "take a partition of unity") of a smooth function $f : \mathbb{R} \to [0, \infty)$ such that $f(x) = 0$ if and only if x is in the standard middle-thirds cantor set $C \subset [0, 1]$ in $\mathbb{R}$.

2. Do the following problems from GP section 2.4 #4, 5, 6, 7, 11

3. Prove that $S^1 \times S^1$ is not simply connected by using the result from GP 2.4 # 4 above, and finding a suitable $Z \subset S^1 \times S^1$ and $f : S^1 \to S^1 \times S^1$. Then use this fact (and a problem from earlier homework) to show that $S^1 \times S^1$ is not diffeomorphic to S^2.