

HW 8 Selected Solution

DF 7.3 #1

Suppose $\phi: 2\mathbb{Z} \rightarrow 3\mathbb{Z}$ was an isomorphism.

Then ϕ would also be an isomorphism of abelian groups $(2\mathbb{Z}, +) \rightarrow (3\mathbb{Z}, +)$

2 generates $2\mathbb{Z}$, so $\phi(2)$ generates $3\mathbb{Z}$

$$\Rightarrow \phi(2) = 3 \text{ or } \phi(2) = -3.$$

$$\text{In } 2\mathbb{Z}, \quad 2+2 = 4 = 2 \cdot 2.$$

$$\text{but } \phi(2) + \phi(2) = 3 + 3 = 6$$

$$\text{or } -3 + -3 = -6$$

$$\text{and } \phi(2)\phi(2) = 9 \neq \phi(2) + \phi(2).$$

So ϕ is not an isomorphism of rings.

DF 7.3 #9

An ideal must be a subring, so the only candidates are: a, b, c, e, and f.

~~of \mathbb{Q}~~

a) is an ideal, since if $f(q) = 0$ for $q \in \mathbb{Q} \cap [0, 1]$
and g is any function, then $g \cdot f(q) = g(q)f(q) = 0$
similarly for $f \cdot g(q)$.

b) is not an ideal, since there exist non polynomial functions (e.g. $\sin(x)$)
and $\sin(x) \cdot 1 = \sin(x)$ which is not a polynomial
 \nwarrow constant polynomial

c) is ~~not~~ not an ideal, since 1 has only a finite number of zeros,
but if f has infinitely many zeros, then $f \cdot 1$ has infinitely
many zeros.

e) is ~~not~~ not an ideal: $\lim_{x \rightarrow 1^-} x-1 = 0$ but if $g(x) = \begin{cases} \frac{1}{x-1} & x \neq 1 \\ 25 & x = 1 \end{cases}$
then $\lim_{x \rightarrow 1^-} (x-1)g \neq 0$ (it doesn't exist!)

f) is not an ideal since $\pi \cdot \sin(x)$ is not a rational linear combination
constant \uparrow function π of $\sin(nx)$ & $\cos(nx)$.

$$2. \phi((a+bi) + (a'+b'i)) = \phi((a+a') + (b+b')i) = \begin{pmatrix} a+a' & -b-b' \\ b+b' & a+a' \end{pmatrix} \\ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} a' & -b' \\ b' & a' \end{pmatrix} = \phi(a+bi) + \phi(a'+b'i)$$

$$\phi((a+bi)(a'+b'i)) = \phi(aa' - bb' + (ab' + ba')i) = \begin{pmatrix} aa' - bb' & -ab' - ba' \\ ab' + ba' & aa' - bb' \end{pmatrix} \\ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a' & -b' \\ b' & a' \end{pmatrix} = \phi(a+bi)\phi(a'+b'i)$$

so ϕ is a ring homomorphism.

$$\text{Ker}(\phi) = \{(a+bi) \in \mathbb{C} \mid \phi(a+bi) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} = \{0\}$$

$$\text{Image: } \phi(\mathbb{C}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \subset M_2(\mathbb{R}).$$

This subring $\phi(\mathbb{C})$ is isomorphic to \mathbb{C} , since $\text{Ker}(\phi) = \{0\}$.

$$3. \text{ Let } I \text{ be the set of matrices } \left\{ \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \mid x_{ij} \in \mathbb{R} \right\}.$$

I is a subgroup, since closed under subtraction and a subring since $\begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} y_{11} \\ \vdots \\ y_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} \\ \vdots \\ x_{n1}y_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix}$

If $A = (a_{ij})$ is another matrix,

$$A \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \text{ has } i\text{-th entry } \sum a_{ij} x_{jk} = \sum a_{ij} x_{j1} \text{ if } k=1, \text{ and } 0 \text{ otherwise} \\ \text{so } A \cdot \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \in I$$

$$\text{But } \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{11} & \dots & x_{11} \\ x_{21} & x_{21} & \dots & x_{21} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n1} & \dots & x_{n1} \end{pmatrix} \notin I \text{ if } x_{11} \neq 0.$$

so I is a left but not a right ideal.

4. Yes, it is an ideal of $\mathbb{R}[x]$ - it is straightforward to check.

DF 7.4 # 10

Let R be a commutative ring and $P \subseteq R$ a prime ideal that contains no zero divisors. We need to show that R contains no zero divisors.

Suppose for contradiction that $ab = 0$ for some $a \neq 0$, $b \neq 0$ in R . Since P is an ideal, $0 \in P$ so $ab \in P$. By definition of prime, either $a \in P$ or $b \in P$. But this contradicts the fact that P has no zero divisors.

15a. Using the result of 14a, b every element of $\mathbb{F}_2[x]/(x^2+x+1)$ is of the form $\overline{p(x)}$ where p has degree 1 or 2.

There are four such polynomials, $0, 1, x$ and $x+1$ in $\mathbb{F}_2[x]$.

6. Suppose for contradiction that $(x^2, 4x) = (p(x))$ in $\mathbb{Z}[x]$.

Since $x^2 \in (p(x))$, ~~then~~ $x^2 = q(x)p(x)$ for some polynomial $q(x) \in \mathbb{Z}[x]$.

So $p(x)$ is equal to ± 1 , $\pm x$ or $\pm x^2$.

$(x^2, 4x) \neq \mathbb{Z}[x]$, since every polynomial in $(x^2, 4x)$ is divisible by x .

So $p(x) \neq \pm 1$. If $p(x) = \pm x^2$, then $(p(x)) = \left\{ \begin{array}{l} \text{polynomials} \\ \text{with no constant} \\ \text{and no } x \text{ term} \end{array} \right\}$
so $4x \notin (p(x))$. So $p(x) \neq \pm x^2$.

If $p(x) = \pm x$, then $x \in (p(x))$. But $x \notin (x^2, 4x)$ since for any $f(x) \in (x^2, 4x)$, the coefficient of x is divisible by 4.

Thus, $(x^2, 4x)$ is not principal.