HW 7, Selected solutions

(solutions to DF problems are at the end)

2. Let R be a ring, $a, b \in R$.

$$0 = 0$$

$$(-a+a)(-b) = a(-b+b) \quad \text{since } (-a+a)=0, \text{ and } 0\cdot(-b)=0 \text{ (proved in class)}$$
$$(-b+b)=0, \ a\cdot 0 = 0$$

$$(-a)(-b) + a(-b) = a(-b) + ab \quad \text{(distributivity)}$$

$$(-a)(-b) = ab \quad \text{(cancellation for additive group)}.$$

3. Let U denote the set of units in R: $U = \{a \in R \mid \exists b \in R \ ab = 1 = ba\}$

Multiplication is a binary operation on U (i.e. U is closed under $\times$) since if $a_1, a_2 \in U$ $\exists b_1, b_2$ s.t. $a_1b_1 = b_1a_1 = a_2b_2 = b_2a_2 = 1$

Then $(a_1a_2)(b_2b_1) = a_1(a_2b_2)b_1 = a_1b_1 = 1$ (associativity).

similarly $(b_2b_1)(a_1a_2) = b_2(b_1a_1)a_2 = 1$

Inverses: if $a \in U$, $\exists b$ s.t. $ab = ba = 1$. ~~[illegible]~~ b is the mult. inverse of a, and $b \in U$ since $ba = ab = 1$.

Associativity: multiplication in R is associative since R is a ring

Identity: $1 \in U$ since $1 \cdot 1 = 1$.

4. a) Let $\alpha = a_1 + b_1\sqrt{D}$, $\beta = a_2 + b_2\sqrt{D}$.

$$N(\alpha\beta) = N(a_1a_2 + b_1b_2D + (a_1b_2 + b_1a_2)\sqrt{D})$$
$$= (a_1a_2 + b_1b_2D)^2 - (a_1b_2 + b_1a_2)^2 D$$
$$= a_1^2a_2^2 + b_1^2b_2^2D^2 + \cancel{2a_1a_2b_1b_2D} - a_1^2b_2^2D - b_1^2a_2^2D - \cancel{2a_1a_2b_1b_2D}$$
$$= a_1^2a_2^2 - a_2^2b_1^2D - a_1^2b_2^2D + b_1^2b_2^2D^2$$

$$N(\alpha)N(\beta) = (a_1^2 - b_1^2D)(a_2^2 - b_2^2D)$$
$$= a_1^2a_2^2 - a_1^2b_2^2D - a_2^2b_1^2D + b_1^2b_2^2D^2$$
$$= N(\alpha\beta)$$

4b) If $\alpha \in \mathbb{Z}[\sqrt{D}]$, then $\alpha \in a + b\sqrt{D}$ for some $a, b \in \mathbb{Z}$.

Since $D \in \mathbb{Z}$, $N(\alpha) = a^2 - b^2 D \in \mathbb{Z}$

c) Assume $D \equiv 1 \pmod 4$.

Let $\alpha = a + b\left(\frac{1+\sqrt{D}}{2}\right)$, $a, b \in \mathbb{Z}$. Then $\alpha = a + \frac{b}{2} + \frac{b}{2}\sqrt{D}$

$$\text{so } N(\alpha) = \left(a + \tfrac{b}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2 D$$
$$= a^2 + ab + \frac{b^2}{4} - \frac{b^2}{4} D$$
$$= a^2 + ab + \frac{1}{4}(1-D)\, b^2$$

since $D \equiv 1 \bmod 4$, $1-D$ is divisible by 4, so $\frac{1}{4}(1-D) \in \mathbb{Z}$, and $N(\alpha) \in \mathbb{Z}$

(case for $D \equiv 2, 3 \bmod 4$ only)

d) Suppose $\alpha \in \mathbb{Z}[\sqrt{D}]$ is a unit. Then $\exists \beta \in \mathbb{Z}[\sqrt{D}]$ such that $\alpha\beta = 1$

by part a, $N(\alpha)N(\beta) = N(\alpha\beta) = N(1) = 1$.

since $N(\alpha)$ and $N(\beta)$ are integers, $N(\alpha) = \pm 1$.

Suppose $N(\alpha) = 1$ and $\alpha = a + b\sqrt{D}$. Then let $\beta = a - b\sqrt{D}$. $\alpha\beta = a^2 - b^2 D = 1 = \beta\alpha$ since $N(\alpha) = 1$

so α is a unit.

DF 7.1 # 3

If u is a unit in S, then there is some $r \in S$ such that $ru = ur = 1_S = 1_R$.

As a counterexample to the converse, 2 is a unit in $\mathbb{Q}$ but not in the subring $\mathbb{Z}$.

DF 7.1 #6

a) Yes, b) Yes

c) No: if $f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ -1 & \text{else} \end{cases}$ and $g(x) = 1$, then $f+g$ is not in the set.

d) No, again not closed under $+$

e) Yes (limit rules!)

f) Secretly this is a problem about <u>fourier series</u> (look it up!) but you can find the answer by using the formulas

$\cos(nx)\cos(mx) = \frac{1}{2}\left(\cos((n+m)x) + \cos((n-m)x)\right)$

$\cos(nx)\sin(mx) = \frac{1}{2}\left(\sin((n+m)x) - \sin((n-m)x)\right)$ etc. to show it is closed under multiplication.

DF 7.1 #15

Suppose for all $a \in R$, $a^2 = a$. Let $x, y \in R$

Then $-(yx) = (-(yx))^2 = (yx)^2$ (by problem 2)

$= yx$

Also $x+y = (x+y)^2 = x^2 + xy + yx + y^2$

$= x + xy + yx + y$

$\Rightarrow \quad 0 = xy + yx$

$\Rightarrow \quad xy = -(yx)$

$= yx$ by the computation above.

DF 7.2 # 8 Let A be a strictly upper triangular matrix in $M_n(S)$.

We will prove as a first step that: if B has all entries zero up to $k-1$ lines above the main diagonal (i.e. $B = (b_{ij})$ and $b_{ij} = 0$ whenever $j - i \leq k$) then AB has all entries 0 up to k lines above the main diagonal.

Proof: The i,j entry of AB is $\sum_{m=1}^{n} a_{i,m} b_{mj}$ when $A = (a_{ij})$

Assume $j - i \leq k+1$. $\sum_{m=1}^{n} a_{im} b_{mj} = \sum_{m=1}^{i} a_{im} b_{mj} + \sum_{m=i+1}^{n} a_{im} b_{mj}$

When $m \leq i$, then $a_{im} = 0$ since A is strictly upper triangular.

If B has all entries $b_{ij} = 0$ whenever $j - i \leq k$,

then, for $m \geq i+1$, $j - m \leq k$, so $b_{mj} = 0$

Thus $\sum a_{im} b_{mj} = \sum 0 \cdot b_{mj} + \sum a_{im} \cdot 0 = 0$, for $j - i \leq k+1$.

Now we show $A^n = 0$.

A has all entries zero except for those a_{ij} s.t. $j - i \leq 1$. (i.e. up to the main diagonal)

By our work above, $A \cdot A$ has all entries zero up to 1 line above the main diagonal.

By induction, assume that A^k has all entries 0 up to $k-1$ lines above the main diagonal

This shows A^n has all entries 0, so $A^n = 0$.

$((A^n)_{ij} = 0$ if $j - i \leq n-1)$