HW 4 Selected solutions

1a. Let H be normal in G. Let $g \in G$.

Then $gHg^{-1} = H$, i.e. $\{ghg^{-1} \mid h \in H\} = H$.

$Hg = \{hg \mid h \in H\}$ by definition.

$= \{hg \mid h \in gHg^{-1}\}$ since $H = gHg^{-1}$

$= \{gh'g^{-1} \cdot g \mid h' \in H\}$ by definition of gHg^{-1}

$= \{gh' \mid h' \in H\}$

$= gH$.

Conversely, suppose that $gH = Hg$ for all $g \in G$. Then, by definition

$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

$= \{xg^{-1} \mid x \in gH\}$ by definition of gH

$= \{xg^{-1} \mid x \in Hg\}$ since $gH = Hg$

$= \{hg \cdot g^{-1} \mid h \in H\}$

$= H$ since $gg^{-1} = e$.

~~SO~~ since this holds for all $g \in G$, H is normal

b. Let $\phi: A \times B \to B$ be defined by $\phi(a, b) = b$.

Claim: ϕ is a homomorphism. Proof: $\phi((a_1, b_1)(a_2, b_2)) = \phi(a_1a_2, b_1b_2) = b_1b_2$

$= \phi(a_1, b_1)\phi(a_2, b_2)$

$Ker(\phi) = \{(a, b) \in A \times B \mid \phi(a, b) = 1_B\} = \{(a, 1_B) \mid a \in A\}$.

Since kernels are normal subgroups, $\{(a, 1_B) \mid a \in A\}$ is a normal subgroup.

2. DF 3.1

#9. $\phi((a+bi)(c+di)) = \phi(ac-bd+(ad+bc)i) = (ac-bd)^2+(ad+cb)^2$
$= a^2c^2+b^2d^2+a^2d^2+b^2c^2$

$\phi(a+bi)\phi(c+di) = (a^2+b^2)(c^2+d^2)$
$= a^2c^2+a^2d^2+b^2c^2+b^2d^2 = \phi((a+bi)(c+di))$

so ϕ is a homomorphism.

$\text{Ker}(\phi) = \{(a+bi) \in \mathbb{C}^\times \mid a^2+b^2\}$ this is a circle of radius 1 in the plane.

Fibers: $\phi^{-1}(r) = \{(a+bi) \in \mathbb{C}^\times \mid a^2+b^2 = r\}$
this is the circle of radius $\sqrt{r}$ in the complex plane.

3. a) $\sigma(x_1,x_2,x_3,x_4) = (x_4, x_3, x_2, x_1)$

"σ moves the k^{th} coordinate to position $\sigma(k)$"

b) $(\sigma\circ\tau)\cdot(x_1,\dots x_n) = (x_{(\sigma\tau)^{-1}(1)}, \dots x_{(\sigma\tau)^{-1}(n)})$
$= (x_{\tau^{-1}\sigma^{-1}(1)}, \dots x_{\tau^{-1}\sigma^{-1}(n)})$

$\sigma\cdot(\tau\cdot(x_1,\dots x_n)) = \sigma\cdot(x_{\tau^{-1}(1)}, \dots x_{\tau^{-1}(n)})$

this is the vector that has $x_{\tau^{-1}(k)}$ in the position $\sigma(k)$

equivalently, with $x_{\tau^{-1}\sigma^{-1}(m)}$ in the position m

$= (x_{\tau^{-1}\sigma^{-1}(m)}, \dots x_{\tau^{-1}\sigma^{-1}(m)})$

4a. Identity $= I$ (check)

Associativity follows from associativity of matrix multiplication:

$$A * (B * C) = A * (CB) = CBA$$
$$= (BA) * C$$
$$= (A * B) * C$$

Inverses: the inverse of A is A^{-1} (usual matrix inverse)

since $A * A^{-1} = A^{-1}A = I = AA^{-1} = A^{-1} * A$.

(Remark: Given any group, one can build a new group by multiplying in the opposite order!)

4b. You need to check that $\phi(ab) = \underbrace{\phi(a) * \phi(b)}_{\text{multiplied in } \mathbb{R}_n LG}$ for all elements a and b in S_3.

For example: in S_3, $(12)(23) = (123)$

$$\phi(123) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\phi(12)\phi(23) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$ which is what you needed to show.

see next page:

4b (continued)

Some hints for shortcuts: (optional)

Option 1: systematic shortcut.

Let $A = \{(12)\ (23)\}$. Then A generates S_3.

Check by hand that for each element $b \in S_3$ and $a \in A$

$\phi(ab) = \phi(a) * \phi(b)$. [this is not so many things to check]

Now given any product xy, write $x = a_1 a_2 \cdots a_k$ where $a_i \in A$

(note that elements of A are their own inverses, so we don't have to worry about inverses)

By what you just checked

$$\begin{aligned}
\phi(xy) &= \phi(a_1 a_2 \cdots a_k y) \\
&= \phi(a_1) * \phi(a_2 \cdots a_k y) \\
&= \phi(a_1) * \phi(a_2) * \phi(a_3 \cdots a_k y) \\
&\ \ \vdots \\
&= \phi(a_1) * \cdots * \phi(a_k) * \phi(y) \\
&= \phi(a_1 a_2) * \phi(a_3) \cdots * \phi(a_k) * \phi(y) \\
&\ \ \vdots \\
&= \phi(a_1 a_2 \cdots a_k) * \phi(y) \\
&= \phi(x) * \phi(y).
\end{aligned}$$

← officially, this should be an inductive argument

Option 2: Ad-hoc shortcuts.

- Use the fact that $\phi(e)$ is the identity matrix, and that $\phi((12))$, $\phi((13))$ & $\phi((23))$ all are matrices of order 2 (they're row-switching matrices!)

Option 3: Explain facts about matrices that do elementary row & column operations to convince me that ϕ is a homomorphism.

5 a. There are $n!$ permutation matrices

Proof: There are n choices of where to put "1" in the first row. Having placed a 1 in the first row, there are $n-1$ choices of where to place a 1 in the second row

[etc, etc... you finish the argument!]

b. The function ϕ is 1-1, and surjective onto the 6 permutation matrices. Therefore it is a bijection. We just showed that it was a homomorphism

c. Example: $(12) \cdot (x_1, x_2, x_3) = (x_2, x_1, x_3)$

$$\phi(12)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_2, x_1, x_3).$$

6. a) The transposition $(i\ i{+}1)$ is the matrix that sends $(x_1, .. x_i, x_{i+1}, \dots x_n)$ to $(x_1, \dots x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots x_n)$

This is the identity matrix with the i^{th} and $i{+}1^{th}$ rows swapped

$$\text{row } i \rightarrow \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \ddots 1 \end{pmatrix}$$

b) There are many ways to explain this, based on what facts you want to use from linear algebra. Anything is OK with me.

c) If $\sigma \in S_n$ can be written as an even number of transpositions, $\sigma = t_1 \cdots t_{2m}$ where t_i is a transposition.

$$\begin{aligned} \phi(\sigma) &= \phi(t_1) * \dots * \phi(t_{2m}) \quad \text{since } \phi \text{ is a homomorphism} \\ &= \phi(t_{2m}) \cdots \phi(t_1) \end{aligned}$$

So $\det(\Phi(\sigma)) = \det(\Phi(t_{2m})) \cdots \det(\Phi(t_1))$
$= (-1)^{2m}$
$= 1$.

Conversely, suppose $\det(\Phi(\sigma)) = 1$.
Since transpositions generate, we can write $\sigma = t_1 \cdots t_k$
where each t_i is a transposition. (or its inverse, but $t_i^{-1} = t_i$)
$\det(\Phi(\sigma)) = (-1)^k$ by the computation above,
so k must be even.