Homework 3 - Selected solutions

1 a) $rAr^{-1} = \{rar^{-1} \mid a \in A\} = \{rrr^{-1}, rsr^{-1}\} = \{r, r^3 s\}$

this is also equal to sr^{-2} or sr^3.

b) For a power of r, $r^k A r^{-k} = \{r, r^{2k}s\}$ and $r^{2k}s \neq s$ for any $k = 1, 2, 3, 4$

so $r^k A r^{-k} \neq A$ and $r^k \notin N_G A$

If $x \in D_{10}$ is not a power of r (and not the identity) we showed last homework that $rx = xr^{-1}$. Equivalently (take the inverse of both sides) $x^{-1}r^{-1} = rx^{-1}$ or $xrx^{-1} = r^{-1}$. Thus, if x is not a power of r, then $r \notin xAx^{-1}$ so $x \notin N_G A$.

It follows that $N_G A = \{e\}$.

2. DF 2.3

\#2 This is similar to what we did in class. Suppose $|x| = |G| = n$

You need to show that $G = \{x, x^2, \dots x^n, e\}$

and that all of these elements are distinct.

For infinite groups: 2 has infinite order in $\mathbb{Z}$, but does not generate

\#3 Claim: $\bar{n}$ generates $\mathbb{Z}/48\mathbb{Z}$ if and only if n is relatively prime to 48.

Proof: if $\gcd(n, 48) = k \neq 1$ then $\frac{48}{k} \cdot \bar{n} \equiv 0 \mod 48$

so $|\bar{n}| = \frac{48}{k}$ in $\mathbb{Z}/48\mathbb{Z}$, so it cannot generate.

if $\gcd(n, 48) = 1$, then for any $m < 48$, $mn \not\equiv 0 \mod 48$

so $|\bar{n}| = 48$, so $\bar{n}$ generates by problem 2.

Thus, the elements that generate $\mathbb{Z}/48\mathbb{Z}$ are

1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47

3. a) The identity is $(1, 0)$

b) Claim: $\mathbb{Z} \times \mathbb{Z}$ is not cyclic

Proof: Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$.

The subgroup of $\mathbb{Z}\times\mathbb{Z}$ generated by (a,b) is

$\{ (na, nb) \mid n \in \mathbb{Z} \}$.

This is not all of $\mathbb{Z} \times \mathbb{Z}$, since if $a \neq 0$ then it does not contain $(0, 1)$

and if $b \neq 0$, it does not contain $(1, 0)$

and if $a = 0 = b$ then the subgroup is just $\{(0,0)\}$.

Thus, no single element generates $\mathbb{Z} \times \mathbb{Z}$, so it is not cyclic.

c) $\mathbb{Z}$ is cyclic, generated by 1.

Suppose $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ was an isomorphism. Then by problem 4 on the last HW, $\phi(1)$ would generate $\mathbb{Z}\times\mathbb{Z}$, so $\mathbb{Z}\times\mathbb{Z}$ would be cyclic!

4. a) Let G be abelian. Then for all $g, h \in G$, $ghg^{-1} = h$.

Thus, for any subgroup H, and any $g \in G$, $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

$= \{h \mid h \in H\} = H$.

So for all g, $gHg^{-1} = H$, and $N_G(H) = G$.

b) Let H be the subgroup of S_4 generated by (12). $H = \{(12), e\}$.

$(13)(12)(13)^{-1} = (13)(12)(13) = (23)$

so $(13)H(13)^{-1} = \{(23), e\} \neq H$.

c) i) $SL_n\mathbb{R}$ is a subgroup

Proof: Let $M, N \in SL_n\mathbb{R}$. $\det(MN) = \det(M)\det(N) = 1 \cdot 1 = 1$

so $MN \in SL_n\mathbb{R}$. Also $\det(M^{-1}) = \frac{1}{\det(M)} = 1$, so $M^{-1} \in SL_n\mathbb{R}$.

4c) (cont.) ii) $SL_n\mathbb{R}$ is normal: Proof: let $M \in GL_n\mathbb{R}$

We will show that $M\, SL_n\mathbb{R}\, M^{-1} = SL_n\mathbb{R}$.

For any $A \in SL_n\mathbb{R}$, $\det(MAM^{-1}) = \det(M)\det(A)\det(M^{-1}) = \det(M)\cdot\frac{1}{\det(M)} = 1$

So $MAM^{-1} \in SL_n\mathbb{R}$. This shows $M\, SL_n\mathbb{R}\, M^{-1} \subset SL_n\mathbb{R}$.

Now given any $B \in SL_n\mathbb{R}$, $M^{-1}BM \in SL_n\mathbb{R}$ by the same computation as above, and $M(M^{-1}BM)M^{-1} = B$. This shows $SL_n\mathbb{R} \subset M\, SL_n\mathbb{R}\, M^{-1}$.

d) Let $\phi : G \to H$ be a homomorphism.

Claim: $\ker(\phi)$ is normal

Proof: Let $g \in G$, and let $a \in \ker(\phi)$. We show $gag^{-1} \in \ker(\phi)$

$\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = \phi(1) = 1_H$

so $gag^{-1} \in \ker(\phi)$ This shows $g\ker(\phi)g^{-1} \subset \ker(\phi)$.

Now for any $b \in \ker(\phi)$, as before, $g^{-1}bg \in \ker(\phi)$ so

$b = g(g^{-1}bg)g^{-1} \in g\ker(\phi)g^{-1}$. This shows $g\ker(\phi)g^{-1} \supset \ker(\phi)$.

Thus $g\ker(\phi)g^{-1} = \ker(\phi)$ for all $g \in G$, and so $N_G(\ker(\phi)) = G$.

e) Since $\det(MN) = \det(M)\det(N)$, det is a homomorphism from $GL_n\mathbb{R}$ to $(\mathbb{R} - \{0\}, \times)$.

$\ker(\det) = \{ M \in GL_n\mathbb{R} \mid \det(M) = 1 \} = SL_n\mathbb{R}$.

Since kernels are normal subgroups, $SL_n\mathbb{R}$ is normal in $GL_n\mathbb{R}$.

DF 1.7 # 3. Let $r, s \in \mathbb{R}$ and let $(x,y) \in \mathbb{R}^2$.

then
$$\begin{aligned} s\cdot(r\cdot(x,y)) &= s\cdot(x+ry,\ y) \\ &= (x+ry+sy,\ y) \\ &= (x+(r+s)y,\ y) \\ &= (r+s)\cdot(x,y) \end{aligned}$$

Also
$$\begin{aligned} 0\cdot(x,y) &= (x+0y,\ y) \\ &= (x,y). \end{aligned}$$

Thus, the axioms for a group action are satisfied.

8. DF 1.7 # 14

In order to satisfy the axioms, we need for all $g_1, g_2 \in G$ and $a \in G$

$g_1 g_2 \cdot a = g_1 \cdot (g_2 \cdot a)$

Since G is non-abelian, there exists g_1 and g_2 such that $g_1 g_2 \neq g_2 g_1$. Let $a = e$.

Then $g_1 g_2 \cdot a =$ ~~[illegible]~~ $e g_1 g_2 = g_1 g_2$

but $g_1 (g_2 \cdot a) = g_1 \cdot (e g_2) = g_2 g_1$

Since $g_1 g_2 \neq g_2 g_1$, the axiom is not satisfied

6 a) Let $(M, \vec{v})$ and $(N, \vec{w})$ be elements of $\mathrm{Aff}(\mathbb{R}^2)$. Let $\vec{x} \in \mathbb{R}^2$.

$$(M, \vec{v}) \cdot ((N, \vec{w}) \cdot \vec{x}) = (M, \vec{v}) \cdot (N\vec{x} + \vec{w}) = M(N\vec{x} + \vec{w}) + \vec{v}$$
$$= MN\vec{x} + M\vec{w} + \vec{v}$$

On the other hand,

$$((M, \vec{v})(N, \vec{w})) \cdot \vec{x} = (MN, M\vec{w} + \vec{v}) \cdot \vec{x} = MNx + M\vec{w} + \vec{v}.$$

Thus, this satisfies axiom 1.

The identity in $\mathrm{Aff}(\mathbb{R}^2)$ is $(I, \vec{0})$.

$(I, \vec{0}) \cdot \vec{x} = I\vec{x} + \vec{0} = \vec{x}$. This shows the action satisfies axiom 2.

b) ~~[illegible]~~ $\mathrm{Stab}((1,0)) = \{(M, \vec{v}) \in \mathrm{Aff}(\mathbb{R}^2) \mid M\binom{1}{0} + \vec{v} = \binom{1}{0}\}$.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\vec{v} = (v_1, v_2)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a + v_1 \\ c + v_2 \end{pmatrix}$$

So $\mathrm{Stab}(\binom{1}{0}) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \in \mathrm{Aff}(\mathbb{R}^2) \;\middle|\; a + v_1 = 1 \text{ and } c + v_2 = 0 \right\}$

It is not obvious to me that this forms a subgroup, but you can check that it's closed under multiplication in $\mathrm{Aff}(\mathbb{R}^2)$ and under inverses!