HW 2, selected solutions

DF 1.2 #2

If x is not a power of r, then $x = sr^k$ for some k.

$rx = rsr^k$

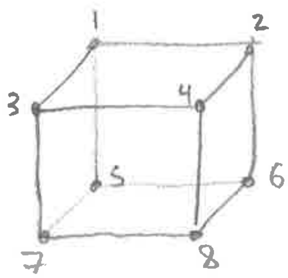
$= (sr^{-1})r^k$ using the relation $rs = sr^{-1}$

$= sr^{k-1}$

$= (sr^k)r^{-1}$

$= xr^{-1}$

#10



A rigid motion of the cube is specified by the new position of the vertices labeled 1, 2, ... 8.

There are 8 choices for the position of 1, then 3 further choices for the position of 2 (it must be a vertex adjacent to 1, but the cube can be rotated so that any adjacent vertex is possible).

There are no rigid motions of the cube in $\mathbb{R}^3$ that fix two adjacent vertices, so the position of other vertices are determined by that of 1 and 2.

This gives $8 \cdot 3 = 24$ elements in the group

DF 1.6 # 2

Let $x \in G$, and let $\phi : G \to H$ be an isomorphism. Suppose $|x| = d$. Then $x^d = e$

so by problem #1, $\phi(x)^d = \phi(x^d) = \phi(e) = e$

and if $j < d$, then $\phi(x)^j = \phi(x^j) \neq \phi(e)$ since ϕ is injective

so $\phi(x)^j \neq e$.

#4. In $\mathbb{R}-\{0\}$, every element has infinite order except for 1 (order 1) and -1 (order 2).

In $\mathbb{C}-\{0\}$, i has order 4 since
$$i^2=-1,\quad i^3=-i,\quad i^4=1$$

By problem #2, $\mathbb{C}-\{0\}$ and $\mathbb{R}-\{0\}$ cannot be isomorphic.

#6 (Hint: $\mathbb{Z}$ is generated by 1, and $\mathbb{Q}$ is not generated by any single element.)

#13 Let $\phi: G \to H$ be a homomorphism.
We use the subgroup criterion.
Suppose $x, y \in \phi(G)$. Then $x=\phi(a)$ and $y=\phi(b)$ for some $a, b \in G$. By problem 1, $y^{-1}=\phi(b^{-1})$ and since ϕ is a homomorphism
$$xy^{-1}=\phi(a)\phi(b)^{-1}=\phi(a)\phi(b^{-1})=\phi(ab^{-1})\in\phi(G).$$

If ϕ is injective, then it is a bijection onto its image, so $\phi: G \to \phi(G)$ is a bijective homomorphism, hence an isomorphism.

DF 2.1 #4. The positive even integers is closed under addition, but not a subgroup.

4. We need to show that any element of H is a product of elements of $\phi(S)$ and their inverses.

Let $h \in H$. Since ϕ is surjective, there is some $g \in G$ so that $\phi(g) = h$. Since S generates G, $g = s_1 \cdot s_2 \cdot \ldots s_n$ where each s_i is either an element of S or $s_i^{-1} \in S$.

$h = \phi(g) = \phi(s_1 s_2 \ldots s_n) = \phi(s_1)\phi(s_2)\ldots\phi(s_n)$

For each i, if $s_i \in S$ then $\phi(s_i) \in \phi(S)$ (by definition)

if $s_i^{-1} \in S$, then $\phi(s_i^{-1}) \in \phi(S)$

and $\phi(s_i^{-1}) = (\phi(s_i))^{-1}$, so $\phi(s_i)$ is the inverse of an element of $\phi(S)$.

Thus, h is a product of elements of $\phi(S)$ and their inverses.

5 a) $(1\,4\,5\,3)$ [note: it's also ok to write $(1453)(2)$, but from now on we'll omit cycles of length 1, so just write (1453)]

b) $(1\,4\,5\,3)^{-1} = (1\,3\,5\,4)$

c) $(1\,3\,5\,4)$ You can either read this off the diagram in question 4, or explain why the inverse of a braid ~~has~~ corresponds to a permutation that is the inverse of the braid's permutation.

d) - This was already done in DF 1.6 # 1 !

e) There are lots of possibilities! here are some:

you can also write them as products of the b_i's for instance b_1^2, $b_2 b_1^2 b_2^{-1}$, b_1^{100000}, $b_2^6 b^2$, ...

7.

a)

$\circ$	e	(123)	(132)	(12)	(13)	(23)
e	e	(123)	(132)	(12)	(13)	(23)
(123)	(123)	(132)	e	(13)	(23)	(12)
(132)	(132)	e	(123)	(23)	(12)	(13)
(12)	(12)	(23)	(13)	e	(132)	(123)
(13)	(13)	(12)	(23)	(123)	e	(132)
(23)	(23)	(13)	(12)	(132)	(123)	e

b)

$\cdot$	e	r	r^2	s	sr^2	sr
e	e	r	r^2	s		
r	r	r^2	e	sr^2		
r^2	r^2	e	r	sr		
s	s	sr	sr^2			
sr^2	sr^2	s	sr			r^2
sr	sr	sr^2	s			

etc... you fill in the rest !!!

c) here is a bijective correspondence between entries

$(123) \longleftrightarrow r$
$(132) \longleftrightarrow r^2$
$(12) \longleftrightarrow s$
$(13) \longleftrightarrow sr^2$
$(23) \longleftrightarrow sr$

Since the multiplication is preserved, this defines an isomorphism.

d) Let G be a group.
Suppose some element $b \in G$ appears twice in the same row of the multiplication table. Then $ax_1 = b$ and $ax_2 = b$ for two <u>different</u> elements x_1 and x_2. This contradicts that equations (i.e. $ax = b$) have unique solutions. The argument for columns is similar.