Homework 11, Selected solutions.

3. Let $\phi : \mathbb{R}[x] \to \mathbb{C}$ be the homomorphism $\phi(p(x)) = p(i)$

We claim that $\ker \phi$ is the ideal generated by x^2+1.

If $p(x) \in (x^2+1)$ then $p(x) = q(x)(x^2+1)$ for some $q(x) \in \mathbb{R}[x]$

so $p(i) = q(i) \cdot 0 = 0$. So $(x^2+1) \subset \ker(\phi)$.

For the reverse containment, suppose $f(x) \in \ker(\phi)$.

Factor f as a product of irreducibles $f(x) = p_1(x)\, p_2(x) \cdots p_k(x)$ in $\mathbb{R}[x]$

By problem from last week's homework, p_k has degree 1 or 2.

Since $\mathbb{C}$ is an integral domain, if $0 = f(i) = p_1(i)\, p_2(i) \cdots p_k(i)$,

then one of the factors $p_k(i) = 0$.

If p_k has degree 1, then $p_k(x) = ax+b$ so $p_k(i) = ai+b \neq 0$. Impossible.

If p_k has degree 2, then p_k is a multiple of x^2+1 (solve for $p_k(i) = 0$)

So we just showed (x^2+1) is a factor of f, so $\ker(\phi) \subset (x^2+1)$.

By the isomorphism theorem for rings $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

4. It is enough to show that $x^k + a^k$ has a root. Let $p(x) = x^k + a^k$.

Since k is odd, $(-a)^k = -a^k$ so $p(-a) = -a^k + a^k = 0$.

5. By the division algorithm, $f(x) = q(x)(x-a)^2 + r(x)$ where $\deg(r(x)) < 2$.

so $r(x) = bx + c$.

Take derivatives: $f'(x) = q'(x)(x-a)^2 + 2(x-a)\,q(x) + \cancel{r'(x)}\, b$

Evaluate at $x=a$ $\quad f'(a) = \cancel{q'(a)(0)} + \cancel{2(0)q(a)} + b$.

so $b = f'(a)$. Also, from the first equation

$$f(a) = \cancel{q(a)(a-a)^2} + r(a) = b \cdot a + c$$

so $c = f(a) - f'(a) \cdot a$

And $r(x) = f'(a)\,x + f(a) - af'(a)$

$= f'(a)(x-a) + f(a)$.

6 . a) This is easy from the description of $\mathbb{Q}(\sqrt{2})$ as $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.
Each element $a + b\sqrt{2}$ has a unique expression as a linear combination of 1 and $\sqrt{2}$, namely $a \cdot 1 + b\sqrt{2}$.

If $a \cdot 1 + b\sqrt{2} = c \cdot 1 + d\sqrt{2}$, then $a - c + (b-d)\sqrt{2} = 0 \Rightarrow a = c$ and $b = d$.

b) Yes $\{1, 3+5\sqrt{2}\}$ is a basis since $\sqrt{2} = \frac{1}{5}(3 + 5\sqrt{2}) + \frac{-3}{5} \cdot 1$ so is a linear combination of 1 & $3+5\sqrt{2}$.

This shows that $\{1, 3+5\sqrt{2}\}$ spans and neither 1 nor $3+5\sqrt{2}$ spans.

$\{3, \sqrt{2}, 5\}$ is not a basis because it has 3 elements.
(or because 3 and 5 are not linearly indep.)

7. $\{a + b\sqrt{2} + c\sqrt{3} \mid a, b, c \in \mathbb{Q}\}$ is not closed under multiplication since $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ cannot be written in this form.

DF 9.4 #3

Let $p(x) = \prod_{i=1}^{n} (x-i) - 1$ in $\mathbb{Z}[x]$.

Suppose for contradiction that $p(x)$ is reducible. Then $p(x) = q(x)\,r(x)$
Since p is monic, degree $q(x) < n$ and degree $r(x) < n$.

If $x = 1, 2, \ldots,$ or n then $p(x) = -1$
so $q(x)\,r(x) = -1$ for $x = 1, 2, \ldots n$.

and either $r(x)$ or $q(x)$ is 1 and the other is -1
This means $q(x) = -r(x)$ for $x = 1, 2, \ldots n$
i.e. $q(x) + r(x) = 0$ for $x = 1, 2, \ldots n$ and so $q+r$ has n different roots. But degree $(q(x) + r(x)) < n$, so $q(x) + r(x) = 0$ (the constant 0 polynomial) Thus $r(x) = -q(x)$ and $p(x) = -(q(x))^2$
But the leading coefficient of $p(x)$ is 1, and the leading coefficient of $-(q(x))^2$ is some negative number. Contradiction.