1. (a) (1 point) Let G be a group and $\phi : G \to H$ a homomorphism. Define the *fiber*

$\phi^{-1}(x) =$ $\{ g \in G \mid \phi(g) = x \}$

(b) (2 points) If ψ is a group homomorphism, and some fiber $\psi^{-1}(x)$ has n elements, is it necessarily true that every other fiber does also? Explain in one sentence. (Do not assume that the groups are finite).

Yes, each fiber is a coset of $\ker(\psi)$ and these all have the same cardinality.

(if $gK = \{gk_1, \dots gk_n\}$ then $hK = \{hk_1, \dots hk_n\}$,)

Unless: if ψ is not surjective, then some fibers might be empty!

2. (a) (2 points) Give an example of a subset with *at most 4 elements* that *generates* the symmetric group S_4.

$\{ (12), (23), (34) \}$ is one example.

(b) (2 points) Give an example of a subgroup H of S_4 such that $|H| = 4$, and H is not cyclic.

$H =$ subgroup generated by (12) and (34).

$= \{ (12), (34), (12)(34), e \}$.

3. (a) (1 point) Let G be a group of order $p^a m$, where p does not divide m. Define a *Sylow p-subgroup* of G.

Any subgroup H such that $|H| = p^\alpha$

(b) (3 points) How many Sylow 3-subgroups does D_{18} have? Describe them explicitly (e.g. list their elements or generators).

$18 = 9 \cdot 2$

A sylow 3-subgroup has 9 elements, there is only one such, it is $\langle r \rangle$.

(c) (4 points) How many Sylow 2-subgroups does D_{18} have? Describe them explicitly. (e.g. list their elements or their generators).

$n_2 \equiv 1 \mod 2$ and divides 9

$\therefore n_2 = 1, 3$ or 9.

$\langle s \rangle$ has 2 elements, as does all its conjugates: $\langle r^k s r^{-k} \rangle = \langle r^{2k} s \rangle$

So we have $\langle s \rangle, \langle rs \rangle, \langle r^2 s \rangle, \ldots \langle r^8 s \rangle$, 9 Sylow 2-subgroups,

(d) (1 point) Does D_{18} have a normal Sylow 2-subgroup? (yes or no)

NO.

4. Let G be an abelian group, and define $\phi : G \to G$ by $\phi(g) = g^2$.

(a) ($\cancel{8}$ 2 points) Show that ϕ is a homomorphism.

$$\phi(gh) = (gh)^2 = ghgh = g^2h^2 \text{ since } G \text{ is abelian}$$
$$= \phi(g)\phi(h)$$

(b) ($\cancel{4}$ 5 points) Now suppose that G is finite of odd order. Show that ϕ (as above) is actually an isomorphism.

Injective: If $\phi(g) = e$, then $g^2 = e$ so $|g| = 1$ or 2. If $|g| = 1$, $g = e$. If $|g| = 2$, then $|G|$ must be divisible by 2, which is false. So $\ker(\phi) = e$, this implies G is injective.

Surjective: If $g \in G$, we want to write $g = h^2$ for some h. Since $|G|$ is odd, $|g|$ is odd, say $2m+1$. Let $h = g^{m+1}$ then

$$h^2 = g^{2m+2} = g^{2m+1} \cdot g = g$$

(c) (2 points) ~~Are either of the above~~ Is part a) true if G is not abelian? Explain or give a counterexample.

Counterexample:

In D_{10}, $(rs)^2 = rsrs = e$

but $r^2s^2 = r^2 \neq e$.

So ϕ is not a homomorphism (and therefore not an isomorphism either!)

5. In this question, ker denotes the kernel of a homomorphism.

Let G, H and J be groups, and let $\alpha : G \to H$ and $\beta : H \to J$ be homomorphisms. You may take it as a fact that the composition $\beta \circ \alpha$ is a homomorphism from G to J

(a) (5 points) ~~Show~~ Prove that $\ker(\alpha)$ is a normal subgroup of $\ker(\beta \circ \alpha)$.

$\ker(\alpha) \subset \ker(\beta \circ \alpha)$ since if $g \in \ker(\alpha)$,

then $\alpha(g) = e$

so $\beta(\alpha(g)) = e$.

It is a subgroup since kernels are always subgroups

(if $\alpha(g) = e$, $\alpha(h) = e$

then $\alpha(gh^{-1}) = \alpha(g)\, \alpha(h^{-1}) = e \cdot e^{-1} = e$.)

Normal: Let $g \in \ker(\alpha)$ and $k \in \ker(\beta \circ \alpha)$.

Then $\alpha(kgk^{-1}) = \alpha(k)\alpha(g)\alpha(k^{-1})$

$= \alpha(k)\, e\, \alpha(k)^{-1}$

$= e$.

6. (10 points) (multiple choice)

Let $G = (\mathbb{R}, +)$ and let A be the set of all continuous functions $\mathbb{R} \to \mathbb{R}$. Which of the following define an *action* of G on A. **Circle all the actions**

a) For $t \in G$ and $f \in A$, let $t \cdot f$ be the function $(t \cdot f)(x) = f(x + t)$
b) Similarly, let $(t \cdot f)(x) = 7x$
c) $(t \cdot f)(x) = f(x) + 2t$
d) $(t \cdot f)(x) = f(tx)$

For the remaining questions, there is only *one* correct answer.
Circle the correct answer.

I. Let G be a finite group, N a normal subgroup of G, and $g \in G$. Then

a) $|gN|$ divides $|g|$
b) $|gN|$ divides $|G|$
c) $|gN|$ is necessarily equal to $|g|$
d) Both a) and b) are true.

II. Let H be the set of 2×2 invertible matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, with $a, b \in \mathbb{R}$.

a) H is a subgroup of $GL_2(\mathbb{R})$
b) $\phi : \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mapsto b$ is a homomorphism from H to $(\mathbb{R}, +)$
c) All of the above
d) None of the above

III. Let G be a group, $N \trianglelefteq G$ and $\phi : G \to G/N$ the homomorphism $\phi(g) = gN$. Let H be a subgroup of G.

a) $\phi(H) \cong H/(H \cap N)$
b) $H \cap N$ is not necessarily a normal subgroup of H
c) $\phi(H)$ is a subgroup of N.
d) None of the above

IV. Let G be a finite group, and assume G acts on a finite set A. Let $|A|$ denote the number of elements in A. Then

a) $|G|$ necessarily divides $|A|$
b) $|A|$ necessarily divides $|G|$
c) A must have at least as many elements of G
d) None of the above