MATH 185-04: Complex Analysis

Final Review

May 5, 2014

- Stereographic projection
  - formula:

$$x = X/(1-Z) \qquad X = 2x/(|z|^2+1)$$
$$y = Y/(1-Z) \qquad Y = 2y/(|z|^2+1)$$
$$Z = (|z|^2-1)/(|z|^2+1).$$

  - circles go to circles and lines
- branch cuts, Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$
- sequences in $\mathbb{C}$, continuity
  - $s_n + t_n \to s + t$, $s_n t_n \to st$, $s_n/t_n \to s/t$, same statement for limits of functions
  - z_n converges iff $\operatorname{Re} z_n$ and $\operatorname{Im} z_n$ converge
  - open/closed subsets, domains, star-shaped domains, boundary, compactness
  - A continuous real-valued function on a compact set attains its maximum.
- complex differentiation

$$f'(z_0) = \frac{df}{dz}(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}.$$

  - analyticity
  - chain rule, product rule, quotient rule
  - Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

  - If $f(z)$ analytic on domain D and $f'(z_0) \neq 0$, then there is a small disk $z_0 \in U \subseteq D$ such that $f(z)$ is 1-1 on U, $V = f(U)$ is open, and f^{-1} is analytic and satisfies $(f^{-1})'(f(z)) = 1/f'(z)$.

- harmonic functions and harmonic conjugate

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)ds + C$$
$$v(B) = \int_{A}^{B} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

- conformal mapping, Fractional linear transformation:

$$w = f(z) = \frac{z - z_0}{z - z_2}\frac{z_1 - z_2}{z_1 - z_0}.$$

  - dilation (rotation), translation, inversion
  - circles in $\mathbb{C}^*$ to circles

- Green's theorem and consequences

$$\int_{\partial D} Pdx + Qdy = \int\int_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dxdy.$$

  - exact and close differentials
  - independence of path (= exact differential)

$$\frac{-ydx + xdy}{x^2 + y^2}$$

- mean value property, maximum principle

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}.$$

  - $u(z)$ real harmonic, $u(z) \leq M$ on D. If $u(z_0) = M$ for $z_0 \in D$, then $u(z) = M$.
  - h bounded complex harmonic, $|h(z)| \leq M$ on D. if $|h(z_0)| = M$, then $h(z)$ constant
  - if $|h(z)| \leq M$ on ∂D, then $|h(z)| \leq M$ on D

- complex line integral
  - ML estimate
  - primitive functions, fundamental theorem of calculus
  - Cauchy's theorem
  - Cauchy integral formula

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw.$$

  - Liouville's theorem
  - $\oint_{|z|=2} \frac{\sin 2z}{z^2(z-\frac{\pi}{2})(z^2+2\pi)} dz$
- power series
  - uniform convergence of analytic functions
  - singular points and radius of convergence

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}} = \lim \frac{|a_k|}{|a_{k+1}|}$$

  - coefficients

$$a_k = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

  - sums and products of power series
  - orders of zeros
  - uniqueness principle and permanence
    * if $f(z) = g(z)$ on a set with a non isolated point, then $f(z) = g(z)$ on D
    * Suppose $F(z, w)$ is independently analytic in z and w. If E is a set with a non isolated point and $F(z, w) = 0$ for all points of E, then $F(z, w) = 0$ on D.
- Laurent decomposition and Laurent series
  - examples using geometric series

- formula for coefficients

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

- classification of isolated singularities
  * removable singularities, $a_k = 0$ for $k < 0$, $f(z)$ bounded near z_0
  * pole of order N, $a_k = 0$ for $k < -N$, $|f(z)| \to \infty$ as $z \to z_0$
  * essential singularities, for every $w_0 \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$ (Casorati-Weierstrauss)
- meromorphic functions and principal part of $f(z)$, partial fractions decomposition

• residue theorem, rules for calculating residues
  - applications to various real integrals using semicircular contours
  - integrals of trigonometric functions using $z = e^{i\theta}$, $\cos z = \frac{z+1/z}{2}$, $\sin z = \frac{z-1/z}{2i}$
  - keyhole contours for functions with branch cuts
  - fractional residue theorem
  - Jordan's lemma $\int_{\Gamma_R} |e^{iz}||dz| < \pi$

• argument principle, logarithmic integral version and argument version

• Rouche's theorem

• open mapping and inverse function theorems

• prime number theorem
  - definitions of gamma and zeta functions, general idea of extending a function using functional equations
  - Chebyshev Θ function
  - Laplace transforms