Math 185 Lecture 4

Midterm 2

April 7, 2014

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. (a) (10 points) Compute $\oint_{|z|=2} \frac{\sin 2z}{z^2} dz$.

Solution: Let $f(z) = \sin 2z$. Then, $f(z)$ is analytic on the disk $|z| < 2$, so by the Cauchy integral formula, we have that

$$f'(0) = \frac{1}{2\pi i} \oint_{|z|=2} \frac{f(z)}{z^2} dz.$$

We can compute $f'(z) = 2\cos 2z$. Therefore,

$$\begin{aligned} \oint_{|z|=2} \frac{\sin 2z}{z^2} dz &= 2\pi i f'(0) \\ &= 2\pi i(2\cos 0) \\ &= 4\pi i. \end{aligned}$$

(b) (15 points) Write $\oint_{|z|=2} \frac{\sin 2z}{z^2(z-\frac{\pi}{2})(z^2+2\pi)} dz$ as a sum of integrals on the boundaries of disks, each disk containing at most one singularity of $\frac{\sin 2z}{z^2(z-\frac{\pi}{2})(z^2+2\pi)}$. You do not have to evaluate the integral.

Solution: Let $g(z) = \frac{\sin 2z}{z^2(z-\frac{\pi}{2})(z^2+2\pi)}$. Then $g(z)$ has singularities at $z = 0$, $z = \frac{\pi}{2}$, and $z = \pm i\sqrt{2\pi}$. Of these, only 0 and $\frac{\pi}{2}$ lie inside the disk $|z| < 2$.
Let D_ϵ be the disk $|z| < 2$ with two disks of radius ϵ, namely $|z| < \epsilon$ and $|z - \frac{\pi}{2}| < \epsilon$, removed. Then, $g(z)$ is analytic on D_ϵ, so by Cauchy's theorem,

$$\oint_{\partial D_\epsilon} g(z) dz = 0.$$

On the other hand, we can also see that

$$\oint_{\partial D_\epsilon} g(z) dz = \oint_{|z|=2} g(z) dz - \oint_{|z|=\epsilon} g(z) dz - \oint_{|z-\frac{\pi}{2}|=\epsilon} g(z) dz.$$

Hence,

$$\oint_{|z|=2} g(z) dz = \oint_{|z|=\epsilon} g(z) dz + \oint_{|z-\frac{\pi}{2}|=\epsilon} g(z) dz,$$

where the two integrals on the right hand side are on boundaries of disks, each disk containing exactly one singularity, $z = 0$ and $z = \frac{\pi}{2}$, respectively.

2. Determine whether the following statements are true or false. No justification is required.

   (a) (5 points) Every analytic function $f(z)$ on a domain D has a power series expansion $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ for each $z_0 \in D$ with a strictly positive radius of convergence.

   TRUE false

   (b) (5 points) Every closed differential on a domain D is exact.

   true **FALSE**

   (c) (5 points) The function $\sin z$ has $2\pi i$ as a period.

   true **FALSE**

   (d) (5 points) There exists a Laurent series for $\frac{e^z \sin z}{z+2}$ on the annulus $1 < |z| < 3$.

   true **FALSE**

   (e) (5 points) Every analytic function on a domain D has a primitive on D.

   true **FALSE**

3. (a) (8 points) Find the power series expansion of $\cosh z$ and $\sinh z$ about $z = 0$, and determine their radii of convergence.

Solution: Recall that $\cosh z = \frac{e^z+e^{-z}}{2}$ and $\sinh z = \frac{e^z-e^{-z}}{2}$. We also know that $(\cosh z)' = \sinh z$ and $(\sinh z)' = \cosh z$.
We can also check that $\cosh 0 = \frac{e^0+e^0}{2} = 1$ and $\sinh 0 = \frac{e^0-e^0}{2} = 0$. From this, we see that the power series for $\cosh z$ centered at $z = 0$ is given by

$$\begin{aligned}\cosh z &= \cosh 0 + \frac{\sinh 0}{1!}z + \frac{\cosh 0}{2!}z^2 + \frac{\sinh 0}{3!}z^3 + \frac{\cosh 0}{4!}z^4 + \dots \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.\end{aligned}$$

Similarly, the power series expansion for $\sinh z$ is

$$\begin{aligned}\sinh z &= \sinh 0 + \frac{\cosh 0}{1!}z + \frac{\sinh 0}{2!}z^2 + \frac{\cosh 0}{3!}z^3 + \frac{\sinh 0}{4!}z^4 + \dots \\ &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.\end{aligned}$$

Both $\sinh z$ and $\cosh z$ are entire, so in particularly have no singularities. This implies by the Corollary on p. 146 that the radius of convergence for both series is ∞.

(b) (6 points) Find the terms up to order six of the power series expansion of

$$f(z) = (\cosh z - 1)\sinh z$$

about $z = 0$.

Solution: From part (a), we have that the power series expansion for $(\cosh z - 1)$ about $z = 0$ is

$$\cosh z - 1 = \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots$$

and the power series expansion for $\sinh z$ is

$$\sinh z = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots.$$

Then, the power series expansion for $f(z)$ is given by

$$\begin{aligned} f(z) &= \left(\frac{1}{2!}\frac{1}{1!}\right) z^3 + \left(\frac{1}{2!}\frac{1}{3!} + \frac{1}{4!}\frac{1}{1!}\right) z^5 + O(z^7) \\ &= \frac{1}{2}z^3 + \frac{1}{8}z^5 + O(z^7), \end{aligned}$$

where $O(z^7)$ denotes terms of order 7 or higher.

(c) (6 points) Determine the radius of convergence of the power series in part (b), and find the order of the zero of $f(z) = (\cosh z - 1)\sinh z$ at $z = 0$.

Solution: The function $f(z)$ has no singularities, so its radius of convergence is ∞. Since the power series about $z = 0$ has no terms of order ≤ 2, the order of the zero at $z = 0$ is 3.

4. (15 points) Suppose $f(z)$ is an entire function such that $|f(z)| \leq \frac{1}{|z|}$ for $|z| > 1$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

Solution: Let $1 > \epsilon > 0$. Then, for $|z| > \frac{1}{\epsilon}$, we have that $|z| > 1$, so that $|f(z)| \leq \frac{1}{|z|} < \epsilon$.

Take $R = \frac{2}{\epsilon}$. Then, the $f(z)$ is analytic on the disk $|z| < R$. On the boundary, $|z| = R > \frac{1}{\epsilon}$, so that $|f(z)| < \epsilon$ if $|z| = R$. By the maximum modulus principle, then $|f(z)| < \epsilon$ for $|z| \leq R$. Also, if $|z| > R$, then $|z| > \frac{1}{\epsilon}$ so that $|f(z)| < \epsilon$. Hence, $|f(z)| < \epsilon$ for all $z \in \mathbb{C}$.

Since this holds for all ϵ such that $0 < \epsilon < 1$, this means that $|f(z)| = 0$. Hence, $f(z) = 0$ for all $z \in \mathbb{C}$.

Solution: By assumption, if $|z| > 1$, then $|f(z)| \leq \frac{1}{|z|} < 1$. In addition, $|f(z)|$ is continuous on the disk $|z| \leq 1$, which is compact. Hence, $|f(z)|$ achieves a maximum, M on $|z| \leq 1$. Let $M' = \max\{M, 1\}$. Then, the above shows that $|f(z)| \leq M'$ for all $z \in \mathbb{C}$. By Liouville's theorem, since $f(z)$ is a bounded entire function, $f(z)$ must be constant, i.e. $f(z) = c$ for some $c \in \mathbb{C}$.

We will now show that $c = 0$. Again, by assumption, we have that if $|z| > 1$, then $|f(z)| = |c| \leq \frac{1}{|z|}$. If we take $|z| \to \infty$, then $\frac{1}{|z|} \to 0$, so that $|c| \leq 0$. This implies $c = 0$, so that $f(z) = 0$ for all z. More specifically, if $1 > \epsilon > 0$, then we can take $z = \frac{2}{\epsilon}$. Then $|z| > 1$, so

$$|f(z)| = |c| \leq \frac{1}{|z|} = \frac{\epsilon}{2} < \epsilon.$$

Since $|c| < \epsilon$ for every $1 > \epsilon > 0$, it must be that $|c| \leq 0$.

5. (15 points) Suppose that $u(z)$ is a real-valued harmonic function on the domain $|z| < 1$. Show that

$$\int_0^{2\pi} [u(re^{i\theta}) - u(0)] \frac{d\theta}{2\pi} = 0,$$

for $0 < r < 1$.

Solution: By the mean value property for harmonic functions,

$$u(0) = \int_0^{2\pi} u(re^{i\theta}) \frac{d\theta}{2\pi},$$

for $0 < r < 1$, since $|z| < r$ lies inside the domain $|z| < 1$. We note that $u(0)$ can be rewritten as

$$u(0) = \int_0^{2\pi} u(0) \frac{d\theta}{2\pi}.$$

We combine this with the statement from the mean value principle to obtain

$$\begin{aligned} 0 &= \int_0^{2\pi} u(re^{i\theta}) \frac{d\theta}{2\pi} - u(0) \\ &= \int_0^{2\pi} u(re^{i\theta}) \frac{d\theta}{2\pi} - \int_0^{2\pi} u(0) \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} [u(re^{i\theta}) - u(0)] \frac{d\theta}{2\pi} = 0. \end{aligned}$$

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Question:	1	2	3	4	5	Total
Points:	25	25	20	15	15	100
Score:						