Math 185 Lecture 4
Midterm 1
February 21, 2014

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. (15 points) Determine, using the definition of differentiability, whether $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = |z - z_0|$ is differentiable or not differentiable at $z = z_0$. Prove your answer.

Solution: We claim that $f(z)$ is not differentiable. To see this, we will find the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ along the real and imaginary directions.

First, we let $z = z_0 + \epsilon$, so that we approach z_0 along the real direction. Then,

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\epsilon\to 0} \frac{|z_0 + \epsilon - z_0| - |z_0 - z_0|}{z_0 + \epsilon - z_0} \\ &= \lim_{\epsilon\to 0} \frac{|\epsilon|}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\epsilon}{\epsilon} \\ &= 1.\end{aligned}$$

Now, if we let $z = z_0 + i\epsilon$, so that we approach z_0 along the imaginary direction, then,

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\epsilon\to 0} \frac{|z_0 + i\epsilon - z_0| - |z_0 - z_0|}{z_0 + i\epsilon - z_0} \\ &= \lim_{\epsilon\to 0} \frac{|i\epsilon|}{i\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\epsilon}{i\epsilon} \\ &= \frac{1}{i} = -i.\end{aligned}$$

Since the limit differs depending on the direction, then $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ does not exist, so f is not differentiable at z_0.

2. Determine whether the following statements are true or false. No justification is required.

   (a) (4 points) Every Cauchy sequence $\{z_n\}$ in $\mathbb{C}$ converges to some $z \in \mathbb{C}$.

   TRUE false

   (b) (4 points) Under stereographic projection, circles on the sphere project to either circles or straight lines in the plane.

   TRUE false

   (c) (4 points) The image of the annulus $1 < |z| < e$ under the principal branch of the logarithm (= Log) is an open rectangle with vertices $i\pi$, $-i\pi$,$1 + i\pi$, and $1 - i\pi$.

   TRUE false

   (d) (4 points) Every analytic function has an analytic inverse.

   true **FALSE**

   (e) (4 points) The image of the disk $|z - i| < 2$ under the fractional linear transformation taking $2 + i$ to -1, $-i$ to 1, and $3i$ to ∞ is the upper half plane $\operatorname{Im} z > 0$.

   true **FALSE**

3. (a) (10 points) Show that $u(x, y) = e^x \cos y$ is harmonic.

Solution: We first compute $\frac{\partial u}{\partial x} = e^x \cos y$, so that $\frac{\partial^2 u}{\partial x^2} = e^x \cos y$.
Also, $\frac{\partial u}{\partial y} = -e^x \sin y$, and $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$.
Hence,

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y + (-e^x \cos y) = 0,$$

so u is harmonic.

(b) (10 points) Find the general form of the harmonic conjugate $v(x, y)$ of $u(x, y)$.

Solution: From the Cauchy-Riemann equations, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y$. Hence, by integrating with respect to y, $v = e^x \sin y + h(x)$, where x is some function of x.
Now, we use the second Cauchy-Riemann equation:

$$\frac{\partial v}{\partial x} = e^x \sin y + h'(x) = -\frac{\partial u}{\partial y} = -(-e^x \sin y).$$

Hence, $h'(x) = 0$, so $h(x) = C$. So $v(x, y) = e^x \sin y + C$.

(c) (10 points) Find an analytic function $f(z)$, written solely in terms of z (i.e. there should be no x's and y's in your formula), such that $u(x, y) = \operatorname{Re} f(z)$.

Solution: From the previous parts, we have that $f(z) = u(x, y) + iv(x, y) = e^x \cos y + ie^x \sin y$ is one possible solution. Using the Euler formula, we can see that

$$f(z) = e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z.$$

Hence, $f(z) = e^z$ satisfies that $\operatorname{Re} f(z) = u(x, y)$.

4. (15 points) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on $\mathbb{C}$ and let $g(z)$ be a fractional linear transformation. Show that if $f'(z_0) \neq 0$ and if $g(f(z_0)) \neq \infty$, then $g \circ f$ is conformal at $z = z_0$.

Solution: We will use the fact that an analytic function h is conformal at z_0 if $h'(z_0) \neq 0$. We first note that $g \circ f$ is a composition of analytic functions, with f analytic at z_0 and g analytic at $f(z_0)$ (since g is analytic on all of $\mathbb{C}$), so $g \circ f$ is analytic at z_0. Next, we note that by the chain rule,

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Since $g(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$, then

$$g'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}.$$

In particular, since $ad - bc \neq 0$, then $g'(z) \neq 0$ as long as $(cz+d)^2 \neq 0$ (in which case $g'(z)$ is undefined). Notice that $g'(z)$ is undefined if and only if $g(z)$ is undefined. We are told that $g(f(z_0)) \neq \infty$, so that $g'(f(z_0)) \neq 0$.

In addition, we are given that $f'(z_0) \neq 0$. Hence, $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0) \neq 0$. By the theorem we quoted at the beginning, then $g \circ f$ is conformal at z_0.

5. (20 points) Suppose f is analytic on a domain D and $|f|$ is constant on D. Show that f is constant on D.

Solution: Since $|f| = \text{constant}$, then we have that $|f|^2 = c$ for some c. We will write $f(z) = u(x,y) + iv(x,y)$, so that $|f|^2 = (u(x,y))^2 + (v(x,y))^2$.

Now, if we differentiate with respect to x,

$$0 = \frac{\partial}{\partial x}|f|^2 = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \tag{1}$$

Differentiating with respect to y yields

$$0 = \frac{\partial}{\partial y}|f|^2 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \tag{2}$$

Since f is analytic, u, v satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Then, substituting and dividing by 2, Equations 1 and 2 become

$$0 = u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}$$
$$0 = u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}.$$

Now, multiply the first equation by u, and add it to v times the second equation, to yield

$$0 = u^{\frac{\partial u}{\partial x}} - uv\frac{\partial u}{\partial y} + uv\frac{\partial u}{\partial y} + v^2\frac{\partial u}{\partial x} = (u^2 + v^2)\frac{\partial u}{\partial x}.$$

Note $u^2 + v^2 = |f|^2 = c$ is constant. If $c = 0$, the $|f|^2 = 0$ implies that $f(z) = 0$ for all z, so f is constant. Otherwise, the above equation implies that $\frac{\partial u}{\partial x} = 0$. By the Cauchy-Riemann equations, we also have that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0.$$

Similarly, we can take $-v$ times the first equation and add it to u times the second equation to yields $0 = (u^2 + v^2)\frac{\partial u}{\partial y}$ so that $\frac{\partial u}{\partial y} = 0$. Hence, $u = \text{constant}$. Again, by the Cauchy-Riemann equations, we also have that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0.$$

Hence, $v = \text{constant}$ as well. Hence, $f = u + iv = \text{constant}$.

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Question:	1	2	3	4	5	Total
Points:	15	20	30	15	20	100
Score:						