Math 185 Lecture 3

Midterm 1

September 29, 2014

Name: ____

- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ____

1. Determine whether the following statements are true or false. No justification is required.

   (a) (5 points) There exists a branch of $\log z$ that is continuous on $\mathbb{C} \setminus \{0\}$.

   true **FALSE**

   (b) (5 points) A continuous, real-valued function on an open subset of $\mathbb{C}$ attains its maximum.

   true **FALSE**

   (c) (5 points) If $\lim_{n\to\infty} \operatorname{Re} z_n$ and $\lim_{n\to\infty} \operatorname{Im} z_n$ exist, then the sequence $\{z_n\}$ converges in $\mathbb{C}$.

   TRUE false

   (d) (5 points) If $u(x, y)$ and $v(x, y)$ are real-valued functions that have partial derivatives of all order, then $f(z) = u(x, y) + iv(x, y)$ is analytic.

   true **FALSE**

   (e) (5 points) Every fractional linear transformation is a composition of dilations, inversions, rotations, and translations.

   TRUE false

2. (a) (10 points) Show that $u(x, y) = xy$ is harmonic.

Solution: We take the partial derivatives as follows:

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial u}{\partial y} = x$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence,

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0.$$

So u is harmonic.

(b) (10 points) Find the general form of the harmonic conjugate $v(x, y)$ of $u(x, y)$, for the function $u(x, y)$ given in part (a).

Solution: We use the Cauchy-Riemann equations to find the conjugate.

$$y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

Integrating with respect to y implies $v = \frac{1}{2}y^2 + h(x)$ for some function $h(x)$. We now use the other Cauchy-Riemann equation,

$$x = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -h'(x).$$

Integrating then shows that $h(x) = -\frac{1}{2}x^2 + C$ for some constant C. This gives the solution

$$v(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + C.$$

3. (20 points) Let $a \in \mathbb{R}$ be fixed and $f(z) = z - \bar{z} + a\bar{z}^2$. Determine the set of points $z \in \mathbb{C}$ at which $f(z)$ is differentiable.

Solution: Note that letting $z = x + iy$,

$$\begin{aligned} f(z) &= (x+iy) - (x-iy) + a(x-iy)^2 \\ &= 2iy + a(x^2 - y^2 - 2ixy) \\ &= a(x^2 - y^2) + i(2y - 2axy). \end{aligned}$$

So we have that $u(x,y) = a(x^2 - y^2)$ and $v(x,y) = 2y - 2axy$. The function f is differential only when the Cauchy-Riemann equations are satisfied, which is

$$\begin{aligned} 2ax = \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 2 - 2ax \\ -2ay = \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = 2ay. \end{aligned}$$

The first equation is satisfied when $x = \frac{1}{2a}$ as long as $a \neq 0$ (when $a = 0$, the equation has no solution). Adding the restriction that $a \neq 0$, then the second equation only has solution when $y = 0$. Hence, $z = x + iy$ must equal $\frac{1}{2a} + i(0) = \frac{1}{2a}$. The Cauchy-Riemann equations are a sufficient and necessary condition for differentiability, so this is the only point at which f is differentiable.

4. (a) (10 points) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an analytic function and that $f(3+i) = i$, $f'(3+i) = 2$, $f(-1) = 1$, $f'(-1) = 5$. Show that $g(z) = [f(z)]^2$ has an inverse $g^{-1} : U \to \mathbb{C}$ that is analytic on an open set U containing -1.

Solution: By the chain rule, $g'(z) = [f(z)^2]' = 2f(z)f'(z)$. Hence, $g'(3+i) = 2f(3+i)f'(3+i) = 2(i)(2) = 4i$. Since $g'(3+i) \neq 0$, there exists an open set U containing $g(3+i) = i^2 = -1$ on which g^{-1} exists and is analytic, by the theorem on p. 51.

(b) (10 points) Find $(g^{-1})'(-1)$, for the function g^{-1} determined in part (a).

Solution: By the theorem on p. 51, $(g^{-1})(g(z)) = \frac{1}{g'(z)}$. Hence,

$$(g^{-1})(-1) = (g^{-1})(g(3+i)) = \frac{1}{g'(3+i)} = \frac{1}{4i} = \frac{-i}{4}.$$

5. (15 points) Let f be the fractional linear transformation mapping 2 to -1, 0 to $-i$ and $1+i$ to i. Find the image of the set $\{z \in \mathbb{C} : |z-1| < 1\}$ under f.

Solution: The set $\{z : |z-1| < 1\}$ is an open disk of radius 1 with center 1. The triple of points 2, -1, and $1+i$ are all points on the boundary of the disk, which is a circle, and they are mapped to the triple -1, $-i$, and i. Since fractional linear transformations map circles and lines to circles and lines, then the image of the circle $|z-1| = 1$ under f must be the circle that passes through -1, $-i$, and i, which is $|z| = 1$.

The disk $|z-1| < 1$ lies on the right as we travel clockwise around the circle from 2 to 0 and $1+i$. Hence, the image must lie on the right as we travel from $f(2) = -1$ to $f(0) = -i$ to $f(1+i) = i$. This is the exterior of the circle $|z| = 1$, so the image is the set $\{z : |z| > 1\}$.

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Question:	1	2	3	4	5	Total
Points:	25	20	20	20	15	100
Score:						