1 The Fourier Transform

So far, we've discussed how to find the Fourier coefficient for a function on $[-\pi, \pi]$. What if we want to take the coefficients for $[-T, T]$? That is we have a function $f(e^{i\pi\theta/T})$. Then, we can use the change of variables $\phi = \pi\theta/T$, we we have $\phi \in [-\pi, \pi]$ and $f(e^{i\phi})$, so we can find the Fourier coefficients as before:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) e^{-ik\phi} d\phi.$$

To solve this in terms of θ, we use the change of variables and find $d\phi = \frac{\pi}{T} d\theta$, so that

$$c_k = \frac{1}{2T} \int_{-T}^{T} f(e^{i\pi\theta/T}) e^{-ik\pi\theta/T} d\theta,$$

where $f(e^{i\pi\theta/T})$ has Fourier series $\sum c_k e^{-ik\pi\theta/T}$.

Here, c_k can be thought of as the component of f that has frequency $\frac{k}{2T}$. If $f : \mathbb{R} \to \mathbb{C}$, then, if we want the component of f with a fixed frequency λ, we take $T = \frac{n}{2\lambda}$ and $k = n$, and let $n \to \infty$.

This yields as the component of f with frequency λ as

$$\lim_{n\to\infty} \frac{\lambda}{n} \int_{n/2\lambda}^{n/2\lambda} f(\theta) e^{-i2\pi\theta\lambda} d\theta.$$

As this scalar factor λ/n goes to 0 as $n \to 0$, we renormalize it by removing the λ/n part. After a change in variables, replacing θ with x, this gives the Fourier transform of f:

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\lambda x} dx.$$

Given the Fourier transform $\hat{f}$, we can reconstruct the function f, under some conditions on f. This is the so-called Fourier inversion theorem, which states that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i2\pi x\lambda} d\lambda.$$

For f being the restriction of a complex analytic function, this is easily proved using the residue theorem.

Theorem 1.1. *Suppose that $f(z)$ is analytic on the strip $-\alpha < \operatorname{Im} z < \alpha$ and that there exists a constant A such that $|f(x+iy)| \leq \frac{A}{1+x^2}$ for all $|y| < \alpha$. Then, $f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i2\pi\lambda x} d\lambda$.*

We first prove a lemma, that lets us rewrite the Fourier transform.

Lemma 1.2. *Suppose that $f(z)$ is analytic on the strip $-\alpha < \operatorname{Im} z < \alpha$ and that there exists a constant A such that $|f(x+iy)| \leq \frac{A}{1+x^2}$ for all $|y| < \alpha$. Then, if $0 < \beta < \alpha$,*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x - i\beta) e^{-i2\pi\lambda(x - i\beta)} dx,$$

for $\lambda > 0$, and

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x - i\beta) e^{-i2\pi\lambda(x + i\beta)} dx,$$

for $\lambda < 0$.

Proof. We will prove the case when $\lambda > 0$. Take the D to be the rectangle with vertices $\pm R$ and $\pm R - i\beta$. Then, by Cauchy's theorem, we have $\int_{\partial D} f(z) e^{-i2\pi\lambda z} dz = 0$.

As $R \to \infty$, the integral from $-R$ to R becomes the Fourier transform, and the integral from $-R - ib$ to $R - ib$ becomes the integral in the lemma. So to show these two quantities are equal, it suffices to show that the integral goes to 0 on the two vertical segments.

Consider $\int_{-R-ib}^{-R} f(z) e^{-i2\pi\lambda z} dz$.

$$\begin{aligned} |\int_{-R-ib} -R f(z) e^{-i2\pi\lambda z} dz| &\leq \int_0^b |f(-R - it) e^{-i2\pi\lambda(-R-it)}| dt \\ &\leq \int_0^b \frac{A}{1+R^2} e^{-2\pi\lambda t} dt. \end{aligned}$$

This is easily seen to go to 0 as $R \to \infty$. A similar calculation works for the vertical segment R to $R - ib$. □

We now prove the inversion theorem.

Proof. To apply the lemma, we need to break up the integral $\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i2\pi\lambda x} dx$ into $\lambda > 0$ and $\lambda < 0$.

$$\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i2\pi\lambda x} d\lambda = \int_{-\infty}^{0} \hat{f}(\lambda) e^{i2\pi\lambda x} d\lambda + \int_{0}^{\infty} \hat{f}(\lambda) e^{i2\pi\lambda x} d\lambda$$

For the second part, we use the lemma to see that

$$\begin{aligned}\int_0^\infty \hat{f}(\lambda)e^{i2\pi\lambda x}d\lambda &= \int_0^\infty \int_{-\infty}^\infty f(x-i\beta)e^{-i2\pi\lambda(u-i\beta)}e^{-2\pi\lambda x}dud\lambda \\ &= \int_{-\infty}^\infty f(u-i\beta)\int_0^\infty e^{-i2\pi\lambda(u-i\beta-x)}dud\lambda dx \\ &= \int_{-\infty}^\infty f(u-i\beta)\int_0^\infty \frac{-1}{-i2\pi(u-i\beta-x)}du \\ &= \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(u-i\beta)}{u-i\beta-x}du \\ &= \frac{1}{2\pi i}\int_{L_1}\frac{f(w)}{w-x}dw.\end{aligned}$$

where L_1 is the real axis shifted down by β, $u-i\beta$.

$$\int_{-\infty}^0 \hat{f}(\lambda)e^{i2\pi\lambda x}d\lambda = -\frac{1}{2\pi i}\int_{L_2}\frac{f(w)}{w-x}dw.$$

where L_2 is the real line shifted up by β. By the Cauchy integral formula, we have that if D_R is the rectangle with vertices $\pm R \pm i\beta$, then

$$f(x) = \frac{1}{2\pi i}\int_{\partial D_R}\frac{f(w)}{w-x}dw.$$

Taking the limit as $R \to \infty$ then breaks up into 4 pieces – the top and bottom pieces are the ones we just computed to be equal to

$$\int_{-\infty}^0 \hat{f}(\lambda)e^{i2\pi\lambda x}d\lambda + \int_0^\infty \hat{f}(\lambda)e^{i2\pi\lambda x}d\lambda = \int_{-\infty}^\infty \hat{f}(\lambda)e^{i2\pi\lambda x}d\lambda.$$

A ML estimate similar to the one from the lemma shows that the integrals along the vertical segments go to 0 as $R \to \infty$, which proves the theorem. □