Math 185 Lecture 3

Final Exam

December 15, 2014

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- You will have **150 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Do not remove or detach the formula sheet from the exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

Stereographic projection:

$$x = X/(1-Z) \qquad X = 2x/(|z|^2+1)$$
$$y = Y/(1-Z) \qquad Y = 2y/(|z|^2+1)$$
$$Z = (|z|^2-1)/(|z|^2+1).$$

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Harmonic conjugate:

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)ds + C$$
$$v(B) = \int_A^B -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Fractional linear transformation:

$$w = f(z) = \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Mean value property:

$$u(z_0) = \int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}.$$

Cauchy integral formula:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}dw.$$

Power series and Laurent series:

$$a_k = \frac{1}{2\pi i}\oint_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta.$$

Residue theorem:

$$\int_{\partial D} f(z)dz = 2\pi i\sum_{j=1}^{m} \mathrm{Res}[f(z),z_j].$$

Argument principle:

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\int_{\partial D} d\arg(f(z)) = N_0 - N_\infty.$$

Inverse function theorem:

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho} \frac{\zeta f'(\zeta)}{f(\zeta)-w}d\zeta, \qquad |w-w_0|<\delta.$$

Fourier series and Fourier transform:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-ik\theta}d\theta \qquad \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\lambda x}dx.$$

1. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) If $u(x, y)$ is a harmonic function on a bounded domain D, then $u(x, y)$ has a harmonic conjugate on D.

   true **FALSE**

   (b) (2 points) The function $\sin z$ is both 2π and $2\pi i$ periodic.

   true **FALSE**

   (c) (2 points) If $f(z)$ is analytic on a domain D, then $f(z)$ has a primitive on D.

   true **FALSE**

   (d) (2 points) If $f(z)$ is meromorphic on D and $\int_{\partial D} \frac{f'(z)}{f(z)} dz = 10\pi i$, then $f(z)$ has 5 zeros in D.

   true **FALSE**

   (e) (2 points) If z_0 is a removable singularity of $f(z)$, then $\text{Res}[f(z), z_0] = 0$.

   TRUE false

   (f) (2 points) If $f : \mathbb{R} \to \mathbb{R}$ is a 2π periodic function, then the Fourier series for f converges uniformly to f.

   true **FALSE**

   (g) (2 points) If $f(z)$ is analytic on the strip $-\alpha < \text{Im}\, z < \alpha$ and there exists a constant A such that $|f(x+iy)| \leq \frac{A}{1+x^2}$ for all $|y| \leq \alpha$. Then, $f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{2\pi i \lambda x} d\lambda$, or in other words, the inverse Fourier transform is the inverse of the Fourier transform.

   TRUE false

   (h) (2 points) If f is analytic on D and U is an open subset of D, then either $f(U)$ is a single point or $f(U)$ is open.

   TRUE false

   (i) (2 points) The image of the circle $|z| = 1$ under the fractional linear transformation sending $1, i, -i$ to $-2+i, i, -1+i$ is the line $y = 1$.

   TRUE false

   (j) (2 points) If $f(z)$ is analytic at z_0, then $f(z)$ is conformal at z_0.

   true **FALSE**

2. (15 points) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \operatorname{Im} z + i \operatorname{Re} z$. Determine the points in $\mathbb{C}$ at which $f(z)$ is differentiable.

Solution: We have that $f(z) = f(x+iy) = \operatorname{Im} z + \operatorname{Re} z = y + ix$. We set $u(x, y) = y$ and $v(x, y) = x$. Then, $f(z)$ is differentiable at $x + iy$ if and only if u, v satisfy the Cauchy-Riemann equations:

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0.$$
$$1 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -1.$$

As the Cauchy-Riemann equations are never satisfied for any values of x or y, then $f(z)$ is not differentiable anywhere in $\mathbb{C}$.

3. (10 points) Suppose that $u(z)$ is a harmonic function on a bounded domain D, $u(z)$ extends continuously to ∂D, and $|u(z) - \sin(z)| \leq M$ for all $z \in \partial D$. Show that $|u(z) - \sin(z)| \leq M$ for all $z \in D$.

Solution: Since $\sin z$ is analytic on $\mathbb{C}$, and every analytic function is harmonic, then $\sin z$ is a (complex-valued) harmonic function on $\mathbb{C}$. In particular, then $\sin z$ is harmonic on $D \subset \mathbb{C}$.

Scalar multiples and sums of harmonic functions are harmonic, so $g(z) = u(z) - \sin z$ is harmonic on D. By the strict maximum principle, since $|u(z) - \sin z| \leq M$ for all $z \in \partial D$, then it follows that $|u(z) - \sin z| \leq M$ for all $z \in D$.

4. (10 points) Find the Laurent series for

$$\frac{4}{(z-i)(z+3i)}$$

centered at 0 that converges at $z = 4$, and determine the annulus on which it converges.

Solution: We begin by finding the partial fraction decomposition

$$\frac{4}{(z-i)(z+3i)} = \frac{A}{z-i} + \frac{B}{z+3i}.$$

Multiplying through by $(z-i)(z+3i)$ yields

$$4 = A(z+3i) + B(z-i).$$

We conclude that $A + B = 0$ and $3iA - iB = 4$, which has solution $A = -i, B = i$. So the partial fraction decomposition becomes

$$\frac{4}{(z-i)(z+3i)} = \frac{-i}{z-i} + \frac{i}{z+3i}.$$

We first find the Laurent series for $\frac{i}{z+3i}$ that converges at $z = 4$. We have that

$$\frac{i}{z+3i} = \frac{i}{z(1+\frac{3i}{z})} = \frac{i}{z}\sum_{n=0}^{\infty}(-\frac{3i}{z})^n,$$

for $|\frac{3i}{z}| < 1 \Rightarrow |z| > 3$. Similarly,

$$\frac{-i}{z-i} = -\frac{i}{z}\frac{1}{1-\frac{i}{z}} = -\frac{i}{z}\sum_{n=0}^{\infty}(\frac{i}{z})^n$$

for $|\frac{i}{z}| \Rightarrow |z| > 1$.

Both series converge at $z = 4$, so the sum converges at $z = 4$, and the annulus of convergence is $|z| > 3$.

$$\begin{aligned}
\frac{4}{(z-i)(z+3i)} &= \frac{-i}{z-i} + \frac{i}{z+3i} \\
&= \frac{i}{z}\sum_{n=0}^{\infty}(-\frac{3i}{z})^n - \frac{i}{z}\sum_{n=0}^{\infty}(\frac{i}{z})^n \\
&= \frac{i}{z}\sum_{n=-\infty}^{0}(-3i)^{-n}z^n - \frac{i}{z}\sum_{n=-\infty}^{0} i^{-n}z^n \\
&= \frac{i}{z}\sum_{n=-\infty}^{0}(\frac{i^n}{3^n} - (-i)^n)z^n \\
&= \sum_{n=-\infty}^{-1}(-\frac{i^n}{3^{n+1}} + (-1)^{n+1}i^n)z^n.
\end{aligned}$$

5. (a) (10 points) Find the singularities of

$$\frac{e^{iz}}{z(\pi^2 - z^2)}.$$

For each isolated singularity, find the residue.

Solution: The numerator is analytic with no zeros, and the denominator has simple zeros at $z = 0, \pm\pi$. Hence, $f(z) = \frac{e^{iz}}{z(\pi^2 - z^2)}$ has simple poles at $z = 0, \pm\pi$. The residues are, using rule 3 and that $[z(\pi^2 - z^2)]' = \pi^2 - 3z^2$:

$$\text{Res}[f(z), 0] = \frac{e^{i0}}{\pi^2 - 3(0)^2} = \frac{1}{\pi^2}$$
$$\text{Res}[f(z), \pi] = \frac{e^{i\pi}}{\pi^2 - 3(\pi)^2} = \frac{1}{2\pi^2}$$
$$\text{Res}[f(z), -\pi] = \frac{e^{-i\pi}}{\pi^2 - 3(-\pi)^2} = \frac{1}{2\pi^2}.$$

(b) (5 points) Let Γ_R be the upper half circle of radius R. Show that

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z(\pi^2-z^2)}dz = 0.$$

Solution: On Γ_R, we have that $\operatorname{Im} z \geq 0$, so

$$|e^{iz}| = |e^{i(\operatorname{Re} z + i\operatorname{Im} z)}| = |e^{-\operatorname{Im} z + i\operatorname{Re} z}| = |e^{-\operatorname{Im} z}| = e^{-\operatorname{Im} z} \leq e^0 = 1.$$

Hence, on Γ_R, for $R > \pi$, the integrand is bounded:

$$|\frac{e^{iz}}{z(\pi^2-z^2)}| \leq \frac{1}{R(R^2-\pi^2)}.$$

The length of Γ_R is πR. Hence, by the ML estimate,

$$|\int_{\Gamma_R}\frac{e^{iz}}{z(\pi^2-z^2)}dz| \leq \frac{1}{R(R^2-\pi^2)}\pi R.$$

Taking the limit as $R \to \infty$, the right hand side becomes 0, so

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z(\pi^2-z^2)}dz = 0.$$

(c) (10 points) Compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)} dx.$$

Solution: Let $D_{R,\epsilon}$ be the half-disk in the upper half plane of radius R indented (by ϵ) at $0, \pm\pi$.

Then $f(z) = \frac{e^{iz}}{z(\pi^2 - z^2)}$ has no singularities in $D_{R,\epsilon}$, so by Cauchy's theorem, $\int_{\partial D_{R,\epsilon}} f(z)dz = 0$. But we also have that

$$\begin{aligned}\int_{\partial D_{R,\epsilon}} f(z)dz = \int_{\Gamma_R} f(z)dz + \int_{-R}^{-\pi-\epsilon} f(z)dz + \int_{\gamma_\epsilon^1} f(z)dz + \int_{-\pi+\epsilon}^{-\epsilon} f(z)dz \\ + \int_{\gamma_\epsilon^2} f(z)dz + \int_{\epsilon}^{\pi-\epsilon} f(z)dz + \int_{\gamma_\epsilon^3} f(z)dz + \int_{\pi+\epsilon}^{R} f(z)dz.\end{aligned}$$

where γ_ϵ^j are the semicircular arcs oriented in the clockwise direction of radius ϵ about $-\pi, 0, \pi$. Taking the limit as $R \to \infty$ and $\epsilon \to 0$ yields, by the fractional residue theorem,

$$0 = \int_{-\infty}^{\infty} \frac{e^{iz}}{z(\pi^2 - z^2)} dz - \pi i \operatorname{Res}[f(z), -\pi] - \pi i \operatorname{Res}[f(z), 0] - \pi i \operatorname{Res}[f(z), \pi].$$

Rearranging and substituting $z = x$ yields

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(\pi^2 - x^2)} dx = \frac{2i}{\pi}.$$

Taking the imaginary part and noting $e^{ix} = \cos x + i \sin x$ then gives that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)} dx = \frac{2}{\pi}.$$

6. (10 points) Suppose $f(z)$ is analytic on a domain D which contains 0 and $\frac{1}{n}$ for each positive integer n, $f(0) = 0$, and $f(\frac{1}{n}) = \frac{1}{n}$ for all positive integers n. Show that $f(z) = z$ for all $z \in D$.

Solution: Let $g(z) = z$. Then, $g(z)$ is analytic on $\mathbb{C}$, so it is analytic on $D \subset \mathbb{C}$. Let $E = \{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$. E has a non-isolated point at 0 since for any $\epsilon > 0$, there exists an $n > \frac{1}{\epsilon}$, which implies that

$$|0 - \frac{1}{n}| = |\frac{1}{n}| < \epsilon.$$

On E, we have that

$$f(\frac{1}{n}) = \frac{1}{n} = g(\frac{1}{n})$$
$$f(0) = 0 = g(0).$$

By the uniqueness principle, as f and g agree on a subset of D with non-isolated points, we have that $f(z) = g(z) = z$ for all $z \in D$.

7. (10 points) Let

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots a_{n-1} z + a_n$$

be a polynomial of degree n. Show that for any $\epsilon > 0$, the roots of $p(z)$ all lie inside the disk $|z| < \max\{1 + \epsilon, |a_1| + |a_2| + \dots + |a_{n-1}| + |a_n| + \epsilon\}$.

Solution: Let $f(z) = z^n$ and $h(z) = a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$. Let $R = \max\{1 + \epsilon, |a_1| + \dots + |a_n|\}$. We have that on the circle of radius R centered at 0,

$$|f(z)| = R^n.$$

On the other hand, since $R > 1$ and $R > |a_1| + \dots + |a_n|$, using the triangle inequality,

$$\begin{aligned} |h(z)| &= |a_1 z^{n-1} + \dots + a_n| \\ &\leq |a_1 z^{n-1}| + |a_2 z^{n-2}| + \dots + |a_n| \\ &\leq |a_1 z^{n-1}| + |a_2 z^{n-1}| + \dots + |a_n z^{n-1}| \\ &= (|a_1| + \dots + |a_n|)|z^{n-1}| \\ &< R(R^{n-1}) = R^n. \end{aligned}$$

Hence, $|f(z)| > |h(z)|$. By Rouche's theorem, then $p(z) = f(z) + h(z)$ has the same number of roots in the disk of radius R as $f(z)$ does. But $f(z)$ has only a single root of multiplicity n at 0. Hence, $p(z)$ has n roots (up to multiplicity) inside the disk of radius R. As $p(z)$ is a polynomial of degree n, $p(z)$ has n roots total, up to multiplicity, so all of its roots are inside the disk of radius $R = \max\{1+\epsilon, |a_1| + \dots + |a_n|\}$.

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Question:	1	2	3	4	5	6	7	Total
Points:	20	15	10	10	25	10	10	100
Score:								