Math 141
Midterm
October 20, 2014

Name: ______________________________

- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. Define the following terms.

   (a) (3 points) smooth manifold

   Solution: A subset $X \subset \mathbb{R}^N$ is a smooth k-manifold if for every $x \in X$, there exists an open subset $U \subset X$, an open subset $V \subset \mathbb{R}^k$, and a map $\phi : V \to U$ that is a diffeomorphism.

   (b) (3 points) local diffeomorphism

   Solution: A smooth map $f : X \to Y$ between two smooth manifolds X and Y is a local diffeomorphism if for every $x \in X$, there exist open sets $x \in U \subset X$ and $f(x) \in V \subset Y$ such that the restriction $f|_U$ is a difffeomorphism from U to V.

   (c) (3 points) transversal (for maps)

   Solution: A smooth map $f : X \to Y$ between smooth manifolds X, Y is transversal to a submanifold $Z \subset Y$ if for every $x \in f^{-1}(Z)$,

   $$\mathrm{Im} df_x + T_{f(x)}(Z) = T_{f(x)}(Y).$$

   (d) (3 points) transversal (for submanifolds)

   Solution: Two submanifolds $X, Z \subset Y$ of a manifold Y are transversal if for every $x \in X \cap Z$,

   $$T_x(X) + T_x(Z) = T_x(Y).$$

2. (10 points) Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Show that if $(X, \mathcal{T})$ is Hausdorff, then the subspace topology on A is Hausdorff.

Solution: Let $x, y \in A$. Since X is Hausdorff, there exist disjoint open subsets $U, V \subset X$ such that $x \in U$ and $y \in V$. Now, $\tilde{U} = U \cap A$ and $\tilde{V} = V \cap A$ are open subset in the subspace topology on A. As U, V are disjoint, so are $\tilde{U}$ and $\tilde{V}$. Moreover, $x \in \tilde{U}$ and $y \in \tilde{V}$. Hence, A is Hausdorff.

3. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^2 y$$

Show that $X = f^{-1}(1)$ is a manifold.

Solution: We wish to show that 1 is a regular value of $f(x, y)$. Consider $df_{(x,y)}$.

$$df_{(x,y)} = \begin{bmatrix} 2xy & x^2 \end{bmatrix}.$$

If $(x, y) \in f^{-1}(1)$, then $x^2 y = 1$, so that $x \neq 0$ and $y \neq 0$. Hence, $2xy \neq 0$, which implies that $df_{(x,y)}$ is surjective at (x, y). In other words, $f(x, y)$ is a submersion at every point of $f^{-1}(x, y)$, which implies that 1 is a regular value.

By the Preimage Theorem, then $f^{-1}(1)$ is a submanifold of $\mathbb{R}^2$.

4. (a) (10 points) Suppose X is a smooth submanifold of a smooth manifold Y. Show that if i is the inclusion $i : X \to Y$, then $di_x : T_x(X) \to T_x(Y)$ is the inclusion on tangent spaces. (Hint: Use the definition of the derivative map for manifolds.)

Solution: We proceed by using the definition of the derivative map for manifolds. Let $x \in X$ and take a parametrization $\phi : U \to V \subset X$ such that $\phi(0) = x$, and a parametrization $\psi : U' \to V' \subset Y$ such that $\psi(0) = x$. Define $h : U \to U'$ by $\psi^{-1} \circ i \circ \phi$. Then, $di_x = d\psi_0 \circ dh_0 \circ (d\phi_0)^{-1}$. But since i is the inclusion $h = \psi^{-1} \circ i \circ \phi = \psi^{-1} \circ \phi$, so $di_x = d\psi_0 \circ d(\psi^{-1} \circ d\phi)_0 \circ (d\phi_0)^{-1} = d\psi_0 \circ (d\psi_0)^{-1} \circ d\phi_0 \circ (d\phi_0)^{-1}$. Then $d\psi_0 \circ (d\psi_0)^{-1}$ is the identity on $T_x(Y)$ and $d\phi_0 \circ (d\phi_0)^{-1}$ is the identity on $T_x(X)$, so the composition is the inclusion of $T_x(X)$ into $T_x(Y)$, as desired.

(b) (8 points) Show that the inclusion $i : X \to Y$ is an embedding.

Solution: The previously posted solution to this question is not correct. Using the definition of embedding given by Guillemin and Pollack, the inclusion may not be an embedding. Full points were awarded for a correct definition of embedding (as defined by Guillemin and Pollack), including a correct definition of proper.

For extra credit, (i) give a counter-example where the inclusion is not an embedding, (ii) give the *weakest condition* on X such that the inclusion is an embedding, and prove it.

The extra credit is worth 2 points each, for a total of 4 points, and due Friday, October 31 at the beginning of class. For this problem, you are not allowed to discuss with anyone else or refer to any outside resources. Any submissions after 9:15am in class will not be accepted, so do not arrive late.

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Question:	1	2	3	4	Total
Points:	12	10	10	18	50
Score:					