Math 141

Final Exam
December 18, 2014

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- You will have **170 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- You are allowed one 8.5"x11" page of notes. No other notes or the textbook are allowed on the exam. No electronic devices or other tools – besides those required for writing – including cellphones, headphones, calculation aids, or non-prescription lenses will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please attach your 8.5"x11" page of notes to the exam before turning it in.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. Define the following terms.

   (a) (3 points) immersion

   Solution: $f : X \to Y$ is an immersion if for every $x \in X$, $df_x : T_x(X) \Rightarrow T_{f(x)}(Y)$ is injective

   (b) (3 points) Morse function

   Solution: A function $f : X \to \mathbb{R}$ is Morse if whenever $x \in X$ is a critical point of f (i.e. $df_x = 0$), then the Hessian $H(f) = (\frac{\partial f_i}{\partial x_j})_{ij}$ is non-singular.

   (c) (3 points) homotopic maps

   Solution: $f, g : X \to Y$ are homotopic if there exists a smooth map (homotopy) $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

   (d) (3 points) $T_x(X)$, when X is a manifold with boundary and $x \in \partial X$

   Solution: Let $\phi : U \subset \mathbb{H}^k \to V \subset X$ be a parametrization near x. Then, $T_x(X) = \operatorname{Im} d\Phi_{\phi^{-1}(x)}$ where Φ is an extension near $\phi^{-1}(x)$ of ϕ to an open subset of $\mathbb{R}^k$

   (e) (3 points) mod 2 winding number

   Solution: For $f : X \to Y$, $W_2(f, z) = \deg_2(\frac{f(x)-z}{|f(x)-z|})$.

2. Determine whether the following statements are true or false. No justification is required.

   (a) (3 points) Every k-dimensional manifold has an embedding into $\mathbb{R}^{2k+1}$.

   TRUE false

   (b) (3 points) If $(X, \mathcal{T})$ is a connected topological space, then for any $x, y \in X$, there exists a continuous path from x to y.

   true **FALSE**

   (c) (3 points) Homotopy is an equivalence relation.

   TRUE false

   (d) (3 points) If $f : S^k \to \mathbb{R}^{k+1}$ is a smooth map whose image does not contain the origin and $f(-x) = -f(x)$ for all $x \in S^k$, then $W_2(f, 0) = 1$.

   TRUE false

   (e) (3 points) Every compact, connected smooth 1-manifold is diffeomorphic to S^1.

   true **FALSE**

   (f) (3 points) If f is a smooth map from the open unit ball $B^n \subset \mathbb{R}^n$ into itself, then there exists a point $x \in B^n$ such that $f(x) = x$.

   true **FALSE**

3. Compute the following. Explain your computations, but a rigorous proof is not required.

(a) (4 points) $\deg_2(f)$, $f : S^1 \to S^1$ is the identity

> **Solution:** $f : S^1 \to S^1$ is a diffeomorphism, so df_x is an isomorphism, and hence for any $y \in S^1$, y is a regular value of f. But $f^{-1}(y) = y$ has a single point, so $\deg_2(f) = 1$.

(b) (4 points) $\deg_2(f)$, $f : S^1 \to \mathbb{R}$ is the projection onto the x-axis

> **Solution:** The projection on to the x-axis misses the point 2, so $f^{-1}(2) = 0$. As 0 is not the image, it is trivially a regular value, so $\deg_2(f) = 0$.

(c) (4 points) $I_2(X, Z)$, where X and Z are circles of unit radius in $\mathbb{R}^2$, X is centered at $(-1, 0)$, and Z is centered at $(1, 0)$.

> **Solution:** X and Z do not intersect transversally, so one must deform X a little bit, say by translating to the left by ϵ, which makes the intersection empty. This gives $I_2(X, Z) = 0$.

4. (10 points) Let $f : H^3 \to \mathbb{R}$ be given by $x^2 + y^2 + xz$. Show that $f^{-1}(1)$ is a manifold with boundary, and determine its boundary.

Solution: Note that $df_{(x,y,z)} = [2x + z, 2y, x]$. If either x or y or non-zero, then the second or third column of $df_{(x,y,z)}$ is non-zero, so the derivative map is surjective onto $T_{f(x,yz)}(\mathbb{R}) = \mathbb{R}$. If both are zero, then $f(x, y, z) = 0$, so $f^{-1}(1)$ is empty. Hence, f is transversal to $\{1\}$.

Now, ∂H^k is the set $\{(x, y, 0) : x, y \in \mathbb{R}\}$. In particular, $z = 0$. So $\partial f : \partial H^k \to \mathbb{R}$ is given by $\partial f(x, y) = x^2+y^2$. We have that $d\partial f_{(x,y)} = [2x, 2y]$ which is again surjective if either x or y is non-zero. This is always the case for $(x, y) \in \partial f^{-1}(1)$, so ∂f is also transversal to $\{1\}$. By the theorem on p. 60, as $\{1\}$ is a zero dimensional manifold without boundary, $f^{-1}(1)$ is a manifold with boundary, and $\partial f^{-1}(1) = f^{-1}(1) \cap \partial H^k = \partial f^{-1}(1)$. But $\partial f^{-1}(1)$ is the set of points $(x, y, 0)$ such that $x^2 + y^2 = 1$, which is the unit circle in the xy-plane.

5. (10 points) Show that $\mathbb{R}^2$ and the cylinder $S^1 \times \mathbb{R}$ are not diffeomorphic.

Solution: Let X, Z be submanifolds of $\mathbb{R}^2$, with X diffeomorphic to S^1 and Z closed. Then, since S^1 has codimension 1 in $\mathbb{R}^2$, the Jordan-Brouwer separation theorem implies that $S^1 = \partial W$ for some compact manifold with boundary W.

By the Boundary Theorem, then $I_2(X, Z) = 0$ for any closed manifold Z.

Now fix $p \in S^1$ and take $X' = S^1 \times \{0\}$ and $Z' = \{p\} \times \mathbb{R}$ to be submanifolds of the cylinder $S^1 \times \mathbb{R}$. It is easy to see that X' and Z' are transversal at their only intersection point $(p, 0)$ since

$$T_{(p,0)}(S^1 \times \mathbb{R}) = T_p(S^1) \times T_0(\mathbb{R})$$

by Exercise 1.2.9 (presented by Roy). But, also by the same exercise,

$$T_{(p,0)}(S^1 \times \{0\}) = T_p(S^1) \times \{0\}$$
$$T_{(p,0)}(\{p\} \times \mathbb{R}) = \{0\} \times T_0(\mathbb{R}).$$

Hence, $T_{(p,0)}(S^1 \times \mathbb{R}) = T_{(p,0)}(S^1 \times \{0\}) + T_{(p,0)}(\{p\} \times \mathbb{R})$.

As the transversal intersection contains one point, $I_2(X', Z') = 1$. But if $f : S^1 \times \mathbb{R} \to \mathbb{R}^2$ is a diffeomorphism, then by the lemma shown in class, $I_2(f(X'), f(Z')) = I_2(X', Z') = 1$. As X' is diffeomorphic to S^1, so is $f(X')$. This contradicts our earlier statement by letting $X = f(X')$ and $Z = f(Z')$. Hence, such a diffeomorphism f cannot exist.

6. (a) (10 points) Let $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ denote the non-zero real numbers. Show that the function $F : \mathbb{R}_* \times \mathbb{R}^3 \to \mathbb{R}^3$, given by $F(t, v) = p + tv$ where $p \in \mathbb{R}^3$ is fixed, is a submersion.

Solution: We can let $p = (p_1, p_2, p_3)$ and $v = (v_1, v_2, v_3)$, so that $F(t, v) = F(t, v_1, v_2, v_3) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$. Then,

$$dF_{(t,v)} = \begin{bmatrix} v_1 & t & 0 & 0 \\ v_2 & 0 & t & 0 \\ v_3 & 0 & 0 & t \end{bmatrix}.$$

In particular, as $t \neq 0$, then $dF_{(t,v)}$ is surjective, as the last three columns form a basis for $\mathbb{R}^3$.

(b) (10 points) Fix $p \in \mathbb{R}^3 \setminus S^2$. Show that almost every line through p intersects $S^2 \subset \mathbb{R}^3$ transversally.

Solution: By part (a), $F \pitchfork Z$ for any submanifold $Z \subset \mathbb{R}^3$, as $\operatorname{Im} dF_{(t,v)} = T_{F(t,v)}(\mathbb{R}^3)$. By the transversality theorem, then $f_v(t) = F(t,v)$ is transversal to Z for almost every $v \in \mathbb{R}^3$.

But $f_v^{-1}(S^2)$ is precisely the set of points t such that $f(t) = p + tv \in S^2$, or in other words, the set of points at which the line $l = \{p + tv : t \in \mathbb{R}\}$ intersects S^2 (we can ignore $t = 0$ since $f(0) = p$ is not in S^2 by assumption). But $T_{p+tv}(l) = \operatorname{span} v = \operatorname{Im} d(f_v)_t$, so this means exactly that $l \pitchfork S^2$, as desired.

So almost every line through p intersects S^2 transversally.

7. (a) (5 points) Let X be a smooth manifold with boundary, and $x \in \partial X$. Show that there exists a smooth non-negative function $f : U \to \mathbb{R}$ on an open subset $U \subset X$ containing x such that $f(z) = 0$ if and only if $z \in U \cap \partial X$. (Hint: Consider $X = H^k$.)

Solution: Let $\phi : V \subset H^k \to U \subset X$ be a parametrization near x. Let x_k be the corresponding coordinate function, which gives for every $z \in U$ the k-th coordinate of $\phi^{-1}(z)$.

We can see that $x_k(z) = 0$ if and only if $z \in \partial U = U \cap \partial X$. Then set $f = x_k$ and U as above. U is an open subset of X by definition of parametrization.

(b) (10 points) Let X be a smooth manifold with boundary. Show that there exists a smooth non-negative function $f : X \to \mathbb{R}$ such that $\partial X = f^{-1}(0)$.

Solution: For each $x \in \partial X$, let U_x, f_x be the open set and function prescribed by part (a). Let $U_0 = \operatorname{Int} X$, which is open in X. Then, $\{U_x\}_{x\in\partial X} \cup \{U_0\}$ is an open cover of X.

Let θ_i be a partition of unity subordinate to the cover. The, for each i, we have that θ_i is identically 0 except in a closed set contained in U_0 or one of the U_x. Define a function f_i as follows: if θ_i is identically 0 except in a closed set contained in U_0, set $f_i : U_0 \to \mathbb{R}$ to be identically equal to 1. If θ_i is identically 0 except in a closed set contained in some U_{x_i}, let f_i be the corresponding function f_x.

We first note that $\theta_i(z)f_i(z)$ can be extended to be smooth on all of X, and not just on U_{x_i}. This is because θ_i is identically zero outside of U_{x_i}, so we can define the product to be identically 0 outside of U_{x_i}, and this gives a smooth function.

Now, let $f(z) = \sum_i \theta_i(z)f_i(z)$. For every $z \in X$, there is an open neighborhood of z on which only finitely many θ_i are non-zero, so the sum is locally a finite sum. Finite sums of smooth functions are smooth, so f is smooth.

For each $x \in \partial X$, $f_i(x) = 0$ for all i, so $f(x) = 0$. On the other hand, if $z \in \operatorname{Int} X$, then note that the condition that $\sum \theta_i(z) = 1$ for all $z \in X$ implies that there is some i for which $\theta_i(z) > 0$. Then, $z \in \operatorname{Int} U_{x_i}$, so $f_i(z) > 0$ as well, so $f(z) > 0$.

Therefore, $\partial X = f^{-1}(0)$.

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Question:	1	2	3	4	5	6	7	Total
Points:	15	18	12	10	10	20	15	100
Score:								