The Alexander ideal (Knot Another Seminar)

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Telling groups apart can be very difficult. The main idea is that we want to have an invariant that is reasonable to calculate which can distinguish non-isomorphic groups. For knot groups, there is an invariant called the Alexander polynomial, and it comes from a certain module that is defined purely group-theoretically.

1 The Alexander module

Given a group G, there is a construction for a series of modules which we will call the Alexander modules of a group. Let $G = G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \dots$ be the derived series for G, where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. The abelianization $G_{ab} = G/G^{(1)}$ of G acts on derived subgroups by conjugation, and so the abelianization $G_{ab}^{(n)} = G^{(n)}/G^{(n+1)}$ is a $G/G^{(n)}$-module with this action.

Definition 1. *The nth* Alexander module *of a group G is $G_{ab}^{(n)}$ as a $\mathbb{Z}[G/G^{(n)}]$-module. The first Alexander module is called* <u>*the*</u> *Alexander module of G.*

From this point, there are a few ways of defining something like an Alexander polynomial. One option is

Definition 2. *The* Alexander ideal *of G is the annihilator ideal* $\mathrm{Ann}_{G_{ab}}(G_{ab}^{(1)})$.

We will discuss some other options later, like the elementary ideals or the order the module.[1]

When G_{ab} is finitely generated, then the structure theorem for finitely generated abelian groups gives an isomorphism

$$G_{ab} = \mathbb{Z}^k \oplus \mathbb{Z}/(r_1) \oplus \cdots \oplus \mathbb{Z}/(r_s)$$

for some $k \geq 0$ and $r_1 \dots, r_s \geq 1$. By choosing generators $t_1, \dots, t_k$ and $x_1, \dots, x_s$ for the free and cyclic components, the group ring can be given as a quotient of a multivariable Laurent polynomial ring

$$\mathbb{Z}[G_{ab}] = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}, x_1, \dots, x_s]/(x_1^{r_1} - 1, \dots, x_s^{r_s} - 1)$$

This is a Noetherian ring, so the Alexander ideal is finitely generated. Unfortunately, this does not make a terribly useful group invariant because (1) the choice of generators for G_{ab} will change the generating set and (2) ideals do not have canonical generating sets. Sometimes we can choose a distinguished set of generators, for instance by a topological consideration such as orientation. The second problem could be solved with Gröbner bases.

Let us take a moment to translate the definitions to algebraic topology. For a topological space X, the abelianization of $\pi_1(X)$ is $H_1(X)$, and for a covering space $p : \widetilde{X} \to X$ corresponding to the commutator subgroup of $\pi_1(X)$, we are saying that the Alexander module for $\pi_1(X)$ is $H_1(\widetilde{X})$ as a $\mathbb{Z}[H_1(X)]$-module. This topological point of view is what will let us calculate.

[1] 5/4/2019 Warning: the annihilator is generated by a divisor of the usual Alexander polynomial (Crowell 1964).

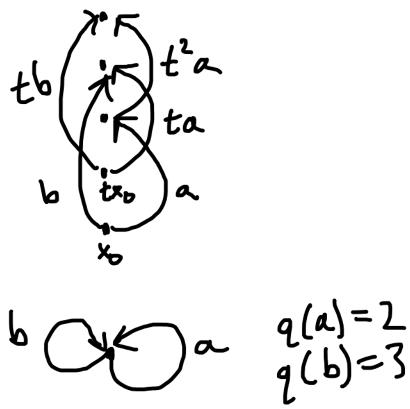

Figure 1: An example of 1-cells a and b lifted according to a map q.

2 Covering spaces

A covering space $p : \widetilde{X} \to X$ is called *normal* if $G' = p_*(\pi_1(\widetilde{X}))$ is a normal subgroup of $G = \pi_1(X)$. For a normal covering space, the fiber over the base point is in one-to-one correspondence with G/G', and the quotient map $q : G \to G/G'$ can be thought of as a way to get "coordinates" for the endpoint of the lift of a loop. If X has a CW structure, then q can be used to create the CW structure for $\widetilde{X}$, essentially by creating a Schreier graph for the cosets of G'.

For simplicity, let us consider the quotient $G/G' = \langle t \rangle$ being a free group with one generator, and suppose X has a single basepoint x_0, finitely many 1-cells $a_1, \ldots, a_n$, and finitely many 2-cells $R_1, \ldots, R_m$. In the cover, the 1-cells can be identified with $t^k x_0$, the 1-cells $t^k a_i$, and the 2-cells $t^k R_i$, for varying k and i. See figure 1.

This means we can represent the chain complex for $\widetilde{X}$

$$C_2(\widetilde{X}) \to C_1(\widetilde{X}) \to C_0(\widetilde{X})$$

by

$$\mathbb{Z}[t, t^{-1}][R_1, \ldots, R_m] \to \mathbb{Z}[t, t^{-1}][a_1, \ldots, a_n] \to \mathbb{Z}[t, t^{-1}]$$

where ∂R_j and ∂a_i are the sum of the components in the lifted boundary, and where the boundary maps are defined to commute with t. The C_1 boundary map is $\partial a_i = (q(a_i) - 1)a_i$, but the C_2 boundary map is more difficult to describe in the general case — examples will make it clear. The first homology group is the Alexander module for $\pi_1(X)$, and the $\mathbb{Z}[t, t^{-1}]$ action is multiplication.

The way forward is to (1) calculate the cycles in C_1, (2) compute the boundaries with respect to this basis, and (3) record this information in a *presentation matrix*. If there are n cycles $z_1, \ldots, z_n$ and m boundaries $b_1, \ldots, b_m$ such that $b_j = \sum_i A_{ij} z_i$, then the $n \times m$ matrix A is the presentation matrix. If $\mathbb{Z}[t, t^{-1}]$ were a PID, then Smith normal form would be sufficient to the Alexander ideal in all cases. Smith normal form happens to work in our examples, anyway, so we will not worry about this yet.

Example. The trefoil knot 2 is a torus knot and has a presentation $\langle a, b | a^3 = b^2 \rangle$. The abelianization is generated by $t = [ba^{-1}]$, so $q(a) = t^2$ and $q(b) = t^3$, thus $C_1 = \mathbb{Z}[t, t^{-1}][a, b]$ and $C_2 = \mathbb{Z}[t, t^{-1}][(1 + t^2 + t^4)a + (1 + t^3)b]$. Cycles are solutions to $f\partial a + g\partial b = 0$, with $f, g \in \mathbb{Z}[t, t^{-1}]$. Since $\partial a = t^2 - 1 = (t-1)(t+1)$ and $\partial b = t^3 - 1 = (t-1)(t^2 + t + 1)$, every solution is a multiple of $f = t^2 + t + 1$ and $g = -(t+1)$, so the cycles are $\mathbb{Z}[t, t^{-1}][(t^2 + t + 1)a - (t+1)b]$. One can write the boundary as $(t^2 - t + 1)((t^2 + t + 1)a - (t+1)b)$, so the presentation matrix is just $[t^2 - t + 1]$. That is, the Alexander module is $\mathbb{Z}[t, t^{-1}]/(t^2 - t + 1)$, so the Alexander ideal is $(t^2 - t + 1)$. It is fairly common to balance this by multiplication by units to get the equivalent $(t - 1 + t^{-1})$.

Example. For (p, q) torus knots in general, with p, q coprime, they have a presentation $\langle a, b | a^p = b^q \rangle$.

Figure 2: The left-handed trefoil knot 3_1.

The same sort of method gives an Alexander ideal of

$$\left(\frac{(1-t^{pq})(1-t)}{(1-t^p)(1-t^q)}\right).$$

In more detail, the boundaries in C_0 are $\partial a = (t^q - 1)x_0$ and $\partial b = (t^p - 1)x_0$, so the C_1 cycles are generated by $\frac{t^p-1}{t-1}a - \frac{t^q-1}{t-1}b$ since these coefficients are cyclotomic polynomials with no common factors. The boundaries in C_1 are generated by $(1 + t^q + t^{2q} + \cdots + t^{(p-1)q})a - (1 + t^p + t^{2p} + \cdots + t^{p(q-1)})b = \frac{1-t^{pq}}{1-t^q}a - \frac{1-t^{pq}}{1-t^p}b$. This is evenly divided by the cycle with quotient $\frac{(1-t^{pq})(1-t)}{(1-t^p)(1-t^q)}$, hence this is the generator for the Alexander ideal.

Example. The group $G = \langle a, b, t | tat^{-1} = b^2, tbt^{-1} = a\rangle$ has a $\mathbb{Z}$ abelianization with $q(a) = q(b) = 1$, $q(t) = t$. The cycles are $\mathbb{Z}[t, t^{-1}][a, b]$, and the boundaries are $ta - 2b$ and $tb - a$. The presentation matrix is

$$\begin{bmatrix} t & -1 \\ -2 & t \end{bmatrix}$$

which has the Smith normal form

$$\begin{bmatrix} 1 & 0 \\ 0 & t^2 - 2 \end{bmatrix}$$

thus the Alexander ideal for G is $(t^2 - 2)$.

Example. Two unlinked circles. The fundamental group of the complement is $G = \mathbb{Z} * \mathbb{Z}$, and the abelianization is $\mathbb{Z}^2$, generated by s and t. The cover's C_1 is $\mathbb{Z}[s, t][a, b] \cong \mathbb{Z}[s, t]$, and since these are all cycles and there are no boundaries, this is the Alexander module. The annihilator is (0).

Example. The Hopf link. The fundamental group of the complement is $\mathbb{Z}^2$, so it is already abelian. There are no cycles, so the Alexander module is 0, hence the annihilator is (1).

3 The Wirtinger presentation

If $K \subset S^3$ is a knot, the *knot group* of K is $\pi_1(S^3 - K)$. Through a straightforward application of the van Kampen theorem, one can use a knot diagram to create a presentation of a knot group. First orient the knot then for each segment in the knot diagram assign a generator representing a right-handed loop. At each crossing, we obtain relations according to the diagram in figure 3.

The abelianization of a knot group is always $\mathbb{Z}$, which is because the relations force the two halves of an understrand to have the same image in the abelianization. The choice of generator in $\mathbb{Z}$ corresponds to the orientation of the knot: if the generator is an image of a segment generator, then it corresponds to the given knot orientation, and otherwise to the opposite.

So, although the knot group is a homeomorphism invariant, the Alexander ideal could, in principle, detect chirality through the choice of generator. A reversed orientation would correspond to the substitution $t \mapsto t^{-1}$ in the ideal. However, for knot groups the polynomials in an Alexander ideal are symmetric Laurent polynomials, so orientation detection fails. This symmetry follows from the following observation. The Knot group

However, we equally could have chosen the relations on the other side of the knot like in figure 3.

By considering the inverses of the generators to be generators of the group, then we have the exact same group presentation but with t^{-1} having the same action as t. Thus polynomials in the Alexander ideal can be given as polynomials in $t + t^{-1}$.

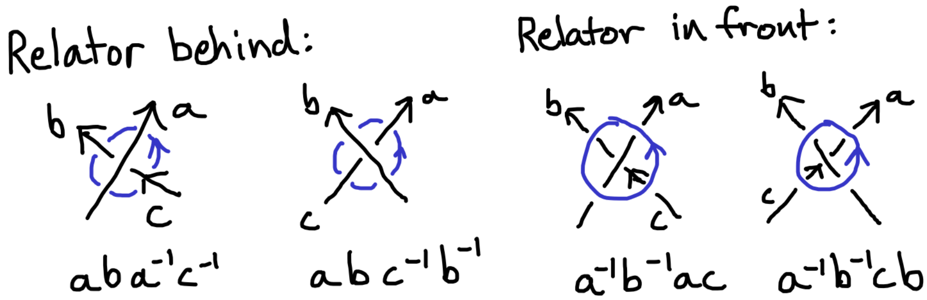

Figure 3: Wirtinger relators behind and in front of the knot.

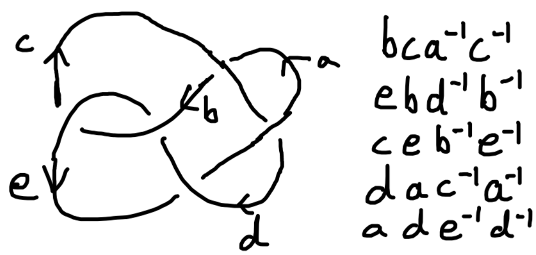

Figure 4: The knot 5_2 with relations.

A consequence to this is that we can sometimes detect when a group is not a knot group. A previous example, like a knot group, had a $\mathbb{Z}$ abelianization, yet its Alexander ideal was not symmetric: $(t^2 - 2)$.

The Wirtinger presentation for a knot has a nice property that, if $a_1 \dots, a_n$ are generators for segments, then in the covering space the lifts go from $t^k x_0$ to $t^{k+1} x_0$, so $a_i - a_n$ is a cycle for all i. This makes it easy to rewrite the boundaries, because it amounts to ignoring a_n in the image.

Example. The knot 5_2 (figure 4) through the calculations yields an Alexander ideal $(2t^2 - 3t + 2)$.

4 Level saturation

In the case of a $\mathbb{Z}$ abelianization, the cycles each have a maximal vertex with respect to the $\mathbb{Z}$-coordinate. We call a particular $\mathbb{Z}$ coordinate a *level*, and we call the level *saturated* if every generator at the level can be written as a sum of generators at a lower level. By reversing the $\mathbb{Z}$ generator, we can consider saturation in the other direction, and a *bidirectionaly saturated* level is a level which is saturated with respect to both generators. Since the cover is normal, saturation is independent of the level, so we call the group bidirectionally saturated if any level is bidirectionally saturated.

This amounts to a criterion for the following lemma:

Lemma 1. *If G is finitely generated and bidirectionally saturated, then $G_{ab}^{(1)}$ is a finitely generated group.*

If $G_{ab}^{(1)}$ is finitely generated, then the Alexander ideal can be calculated by computing the matrix of the t action and then computing the minimal polynomial of that matrix. Thus, in such a case the Alexander ideal generator is monic.

Conversely, if the Alexander ideal is a principal ideal and both the leading and constant terms of the generator are ± 1, then $G_{ab}^{(1)}$ is finitely generated. This is because the generator gives a proof of bidirectional saturation.

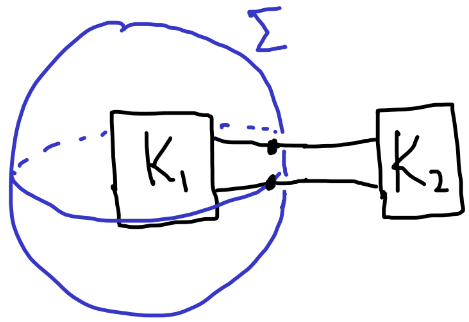

Figure 5: Knot sum $K_1 + K_2$ with sphere Σ.

4.1 Fibered knots

A *fibered knot* is a knot K whose complement $S^3 - K$ is a fiber bundle over S^1 by Seifert surfaces. Such a fiber bundle gives a monodromy action coincident with t, so the annihilator must be generated by a monic polynomial.

A partial converse is given by Stallings.

Theorem 1 (Fibration theorem, Stallings 1962)**.** *Given a compact irreducible* 3*-manifold* M*, a finitely generated group* G *not isomorphic to* $\mathbb{Z}/2\mathbb{Z}$*, and a short exact sequence* $1 \to G \to \pi_1(M) \to \mathbb{Z} \to 1$*, then* M *fibers over* S^1*.*

This applies since

Definition 3. *A* 3*-manifold is* irreducible *if any smooth sphere bounds a ball.*

Lemma 2 (Alexander's lemma)**.** *Up to isotopy, there is a unique PL/smooth embedding of* S^2 *into* S^3*.*

In particular, this implies that if the commutator subgroup of $\pi_1(S^3 - K)$ is finitely generated, then not only is the Alexander ideal generator monic, but K is a fibered knot.

Example. 5_2 is has a non-monic generator, so it is not a fibered knot.

The homology group being finitely generated is not enough for being a fibered knot.

- For ≤ 10 crossings, monic is equivalent to being fibered.
- For 11 crossings, there is $11n_{73}$, which has a nontrivial monic polynomial but is not a fibered knot.
- (Hirasawa). For 11 crossings, monic and knot genus matches degree is equivalent to being fibered.

The saturation condition can be adjusted to deal with commutator subgroups. In fact, if the Wirtinger presentation saturates level 1 at the homotopy level, then it is a fibered knot. Torus knots are examples.

5 Knot sums

Given two knots K_1, K_2, the Alexander ideal of the sum $K_1 + K_2$ is the product of the ideals.

A preliminary fact is that the knot group is isomorphic to the complement of the knot in a ball with one point of the knot at the boundary of the ball. This can be seen by performing an inversion of S^3 through a point inside the knot. The tangent point can be split into two separate strands. Then the knot sum amounts to identifying the boundaries of the knot's respective balls.

Let Σ be a sphere containing the K_1 part of the sum and which intersects the sum in two points, as in figure 5. The van Kampen theorem says that $\pi_1(K_1 + K_2) = \pi_1(K_1) *_{\pi_1(\Sigma)} \pi_1(K_2)$, where the amalgamation in particular identifies the generators for the two strands. By taking the Wirtinger presentation and choosing

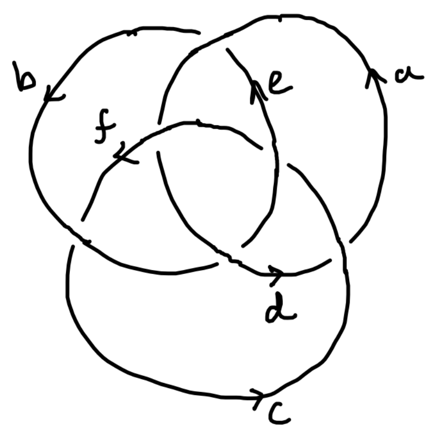

Figure 6: The Borromean links.

these identified strands as the ones to remove in the basis change for computing cycles, the presentation matrix looks like

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

so the annihilator is the product of the annihilators.

This does not mean that an irreducible generator implies the knot is a prime knot, since there are knots whose Alexander ideal is (1). However, it is true for knots with 10 or fewer crossings.

6 Elementary ideals and order ideals

One annoyance with the theory is that $\mathbb{Z}[t, t^{-1}]$ is not a principal ideal domain. We will discuss two corrections.

The first is to tensor everything with $\mathbb{Q}$, making $\mathbb{Q}[t, t^{-1}]$ a principal ideal domain, and the generator is a kind of *Alexander polynomial*. Using annihilators is not the usual definition, however, but instead the *order ideal* is used. Since the $\mathbb{Q}$-tensored Alexander module is a finitely generated module over a principal ideal domain, it is a direct sum of free and cyclic $\mathbb{Q}[t, t^{-1}]$-modules. The order ideal is the product of the orders of each component. This is well-defined and does not depend on the particular decomposition.

The order is also the product of the diagonal after taking the presentation matrix to Smith normal form, which is well-defined over a principal ideal domain. This can also be computed by taking the ideal generated by all of the $n \times n$ minors, given that n is the number of generators for the Alexander module. The order is to the characteristic polynomial as the annihilator is to the minimal polynomial.

The *first elementary ideal* is the ideal generated by the $n \times n$ minors of the presentation matrix when the module is still over $\mathbb{Z}[t, t^{-1}]$. Elementary ideals are principal for knot groups, so we get a well-defined generator up to multiplication by a unit, which is the traditional *Alexander polynomial*.

7 Links

For links, the orientation of component knots can matter.

If links are *split links*, in that there is a separating sphere, then the Alexander ideal is (0). This is because the fundamental group is a free product and the Alexander ideal will have infinite cyclic components.

Example. The Borromean links (figure 6) have 8 choices for the orientations, yet the annihilator is always $(s-1, t-1, u-1)$. Link 4^2_1 has 4 choices, giving two ideals $(t+u^{-1})$ and $(t+u)$.

References

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