A TQFT approach to topological graph polynomials

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University of Virginia Geometry seminar

* Overview

My motivation: Fendley, Krushkal & Agol used tensor category techniques and the Temperley-Lieb alg. to reprove & study Tutte's "golden identity" for the flow poly.

Can universal contraction-deletion graph invariants be studied in a similar way?

A problem: identity elements

Main idea: Frobenius algebras!

Outline:

- Tutte poly (graphs)
- Bollobás-Riordan poly (ribbon graphs)
- Krushkal poly (surface graphs)

* Graph invariants

ex Chromatic poly [Birkhoff 1912; Whitney 1932]

$\chi_G(\lambda)$ = # ways to color vertices with λ colors s.t. adjacent vertices are colored distinct

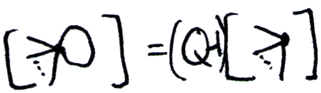
↑non-bridge "deletion – contraction"

 $= 0$ $[G \amalg \bullet] = \lambda [G]$

ex Flow poly [Tutte 1947]

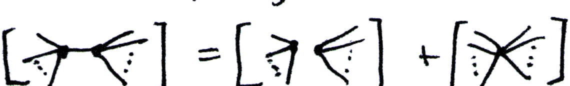
A : abelian group of order Q

$F_G(Q)$ = # nowhere-zero 1-cycles in $C_1(G; A)$

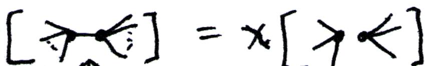
 $[G \amalg \bullet] = [G]$

* Tutte-Whitney polynomial (1954)

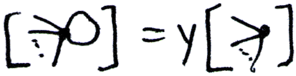
$$T_G(x,y) = \sum_{\substack{H \subseteq G \\ \text{(spanning)}}} (x-1)^{b_0(H) - b_0(G)} (y-1)^{b_1(H)}$$



↑non-bridge

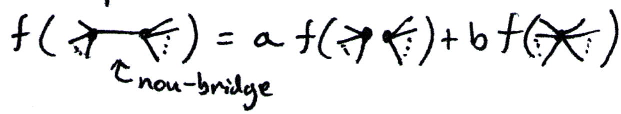


↑bridge

 $[G_1 \amalg G_2] = [G_1][G_2]$ $[\bullet] =$

<u>Thm</u> [Oxley & Welsh 1979, others] R ring,

If f : Graphs $\to R$ satisfies

1) 

2) $f(G_1 \amalg G_2) = f(G_1)f(G_2) = f(G_1 \vee G_2)$

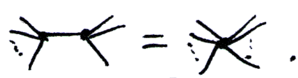
then $f(G) = a^{b_1(G)}\, b^{|V|-b_0(G)}\, T_G\left(\frac{f(\bullet\!\!-\!\!\bullet)}{b}, \frac{f(\circ)}{a}\right)$.

<u>ex</u> $\lambda^{-b_0(G)}\chi_G(\lambda) = (-1)^{|V|-b_0(G)}\,T_G(1-\lambda, 0)$

$F_G(Q) = (-1)^{b_1(G)}\,T_G(0, 1-Q)$

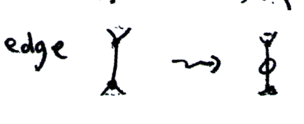
* A 1-D TQFT

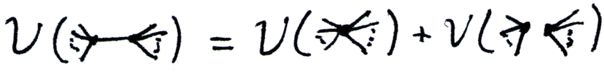
Let $\mathcal{G}^R$ be the following symmetric monoidal category (R a ring)

- Objects: finite sets of points (0-cells)
- Morphisms: $\mathcal{G}^R(A,B)$ is R-linear combinations of homeo. classes of 1-complexes G with $A \amalg B \subseteq G^{(0)}$, modulo .
- Composition: $G_1 \in \mathcal{G}^R(A,B)$, $G_2 \in \mathcal{G}^R(B,C)$, $G_2 \circ G_1 = G_2 \amalg_B G_1$

Let $\phi = | + \dot{|} \in \mathcal{G}^R(\cdot,\cdot)$

define $\mathcal{V}$: Graphs $\longrightarrow \mathcal{G}^R$ by



Then 

<u>note</u> $V(G)$ is a poly in bouquets ()
(this is Tutte's V-poly, 1947)

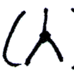
<u>note</u> $V($  $) = V($  $)$

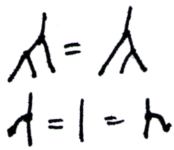
Consider a symm. monoidal functor

$$Z : \mathcal{G}^R \longrightarrow R\text{-Mod}$$

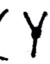
<u>Prop</u> Such Z correspond to commutative Frobenius algebras over R.

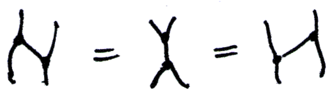
<u>Pf</u> Let $A = Z(\bullet)$

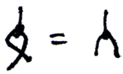
$\mu = Z($  $)$, $A \otimes A \to A$ $\quad i = Z($  $)$, $R \to A$



sim for

$\Delta = Z($  $)$, $A \to A \otimes A$ $\quad \varepsilon = Z($  $)$, $A \to R$

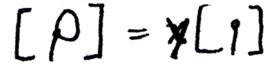
and  so Frob. alg.

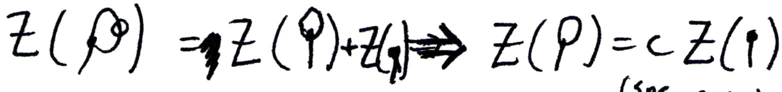
also 

$\blacksquare$

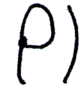
Suppose R a field, char $=0$, $\bar{R} = R$.

Tutte has

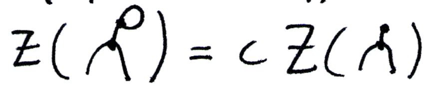




(sps $c \neq 0$)

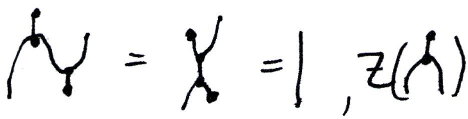
Note $Z($  $) = a \mapsto tr_A(b \mapsto \mu(a \otimes b))$

so



is trace form

Since



is symm. nondegen. bilinear form.

$\Rightarrow$ A is a semisimple algebra

Commutative $\Rightarrow$ $A = \bigoplus_{i=1}^{N} R$

Can calculate

$Z(G) = Z($  $) = Nc^{n-1}$

(G conn.; $b_1(G)$ loops)

Hence $Z \circ U$: Graphs $\to$ R-Mod

is $\quad G \mapsto \sum_{H \subseteq G} (Nc^{-1})^{b_0(H)} c^{b_1(H)}$

$= (Nc^{-1}+1)^{b_0(G)} T_G(Nc^{-1}+1, c+1)$

ex $A = \mathbb{C}[\mathbb{Z}/n\mathbb{Z}] = \mathbb{C}[x]/(x^n - 1)$

with $= x^i \mapsto [x^i = 1]$

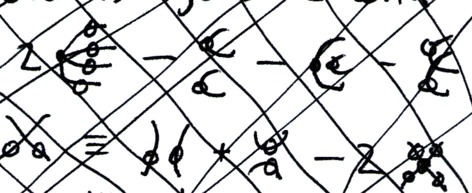
Then $= 1$ and $= n$

With $\phi = 1 -$ (idempotent), get flow poly for $\mathbb{Z}/n\mathbb{Z}$ flows

ex There is pairing $\mathcal{G}^c(4,0) \otimes \mathcal{G}^c(□,Q) \to \mathbb{C}$

$G_1 \otimes G_2 \mapsto F_{G_2 G_1}(Q)$

At $Q=3$, is degenerate with



in ker

so

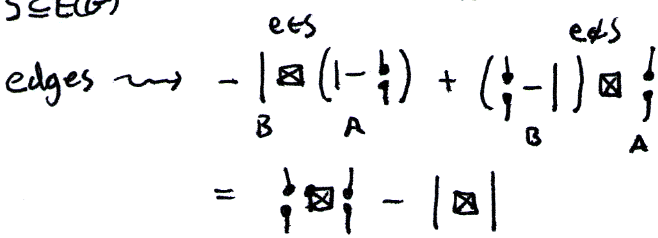
is additional local linear relation.

ex $B = \bigoplus_{i=1}^{m} \mathbb{C}$ $= e_i \mapsto 1$ $= m$ & $=$

with $\phi = - 1$ gives chromatic poly

ex [Kook, Reiner, Stanton 1997]

$$\sum_{S \subseteq E(G)} (-1)^{|S|} \chi_{G/S}(m) F_{G|S}(n) = (-1)^{|V|} (-m)^{b_0(G)} T_G(1-m, 1-n)$$



$$= \sum_{H \subseteq G} (-1)^{|E(H)|} m^{b_0(H)} n^{b_1(H)}$$

ex $B = \bigoplus_{i=1}^{m} \mathbb{C}$ has $S_m \circlearrowright B$ action

Frob. alg. diagrams live in full subcategory of $\mathrm{Rep}(S_m)$ gen. by B.

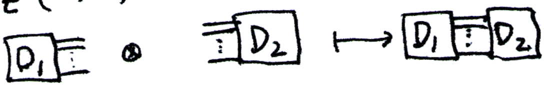
Can define this category for generic m:

partition cat. P_t is category of diagrams with $\bullet = t$, and $P_m(a,b) \twoheadrightarrow \mathrm{Hom}_{S_m}(B^{\otimes a}, B^{\otimes b})$

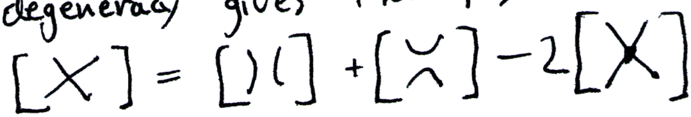
Its pseudo-abelian envelope is Deligne's partition category $\mathrm{Rep}(S_t)$, semisimple for $t \in \mathbb{C} - \mathbb{Z}_{\geq 0}$

Else get additional local linear relations

ex Pairings $P_t(n,0) \otimes P_t(0,n) \to \mathbb{C}$, $t \in \mathbb{C}$



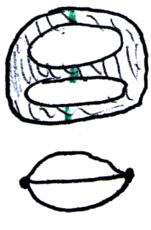
@ $t=3$, degeneracy gives Flow poly relation



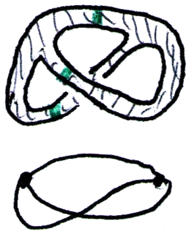
* Bollobás–Riordan poly

def A <u>ribbon graph</u> is a compact ori. sfc Σ with $\partial\Sigma \neq \emptyset$ along with a set of disj. properly emb. arcs $\alpha_1, \dots, \alpha_n$ s.t. $\Sigma - \bigcup \alpha_i$ is a disj. union of disks

<u>ex</u>



vs

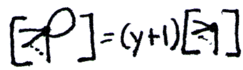


<u>note</u> not emb. in S^3!

<u>def</u> (B&R 2001) G a ribbon graph

$$BR_G(X, y, z) = \sum_{H \subseteq G} (x-1)^{b_0(H) - b_0(G)} \, y^{b_1(H)} \, z^{2g(H)}$$

<u>note</u> $T_G(x, y) = BR_G(x, y-1, 1)$



Has same rules for non-loop edges. $= (y+1)$ interlaced circles

$\rightsquigarrow$ reduce to wedges of interlaced circles

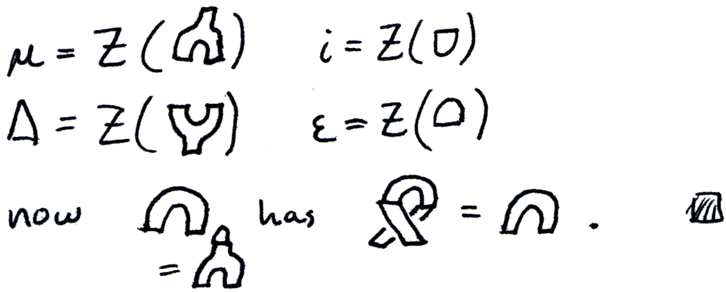
<u>Def</u> R-ring $\mathcal{R}^R$ is symm. monoidal category where

- Objects: disj. unions of ori. closed intervals
- Morphisms: $\mathcal{R}^R(I, J)$ is R-lin. combs of homeo$^+$ classes of compact ori. sfcs with corners Σ with $\partial\Sigma \neq \emptyset$ and $I \amalg J \subsetneq \partial\Sigma$.
- Composition is gluing along shared ∂

"open strings" (See also [Landa, Pfeiffer 2008])

Prop Symm. monoidal functors $Z: \mathcal{R}^R \to R\text{-Mod}$ corr. to symmetric Frobenius algebras

Pf



Let $\square = \square + \square$ and define ν: Ribbongraphs $\to \mathcal{R}^R$.

Again, $Z(\ ) = c\, Z(\ )$

$\Rightarrow$ A is semisimple if $R=\bar{R}$ field char $=0$

Artin-Wedderburn $\Rightarrow A \approx \bigoplus_{i=1}^{N} Mat_{n_i}(R)$

and $Z(\ ) = c^{-1} \sum_{i=1}^{N} n_i \, tr_i$

Can calculate

$$Z(G)_{\text{conn.}} = Z(\ \underbrace{\quad}_{b_1(G)-2g(G)} \ \underbrace{\quad}_{g(G)}\ ) = c^{b_1(G)-1} \sum_{i=1}^{N} n_i^{2-2g(G)}$$

$Z \circ \nu$: Ribbongraphs $\longrightarrow$ R-Mod

has $G \longmapsto \sum_{H \subseteq G} c^{b_1(H) - b_0(H)} \prod_{F \in \pi_0(H)} X_{g(F)}$

with $X_g = \sum_{i=1}^{N} n_i^{2-2g}$. Newton polys

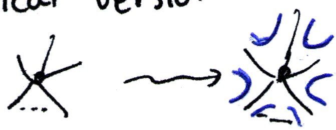
Like [Goodall et al 2016]

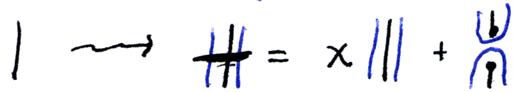
<u>ex</u> $A = \mathbb{C}[\mathbb{Z}/n\mathbb{Z}] \otimes \mathrm{Mat}_m\mathbb{C} \cong \bigoplus_{i=1}^{n} \mathrm{Mat}_m\mathbb{C}$

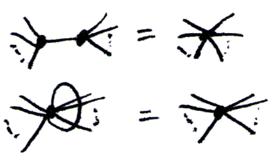
has $\chi_g = n\, m^{2-2g}$

gives $(m^2 n c^{-1})^{b_0(G)} BR_G(m^2 n c^{-1}+1, c, m^{-1})$

* A graphical version

vtx 

edge 

with 1), 2)  } b_0

3) $\bullet = n$

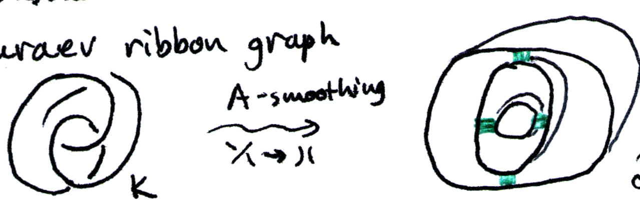
4)  $= m$ } $b_0(\partial)$

Gives $BR'_G(x,n,m) = \sum_{H \subseteq E(G)} x^{|E(G)|} n^{b_0(H)} m^{b_0(\partial H)}$

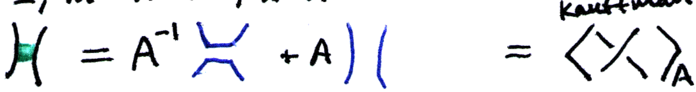
$= x^{|V|}(x^{-1}nm)^{b_0(G)} BR_G(x^{-1}nm+1, xm, m^{-2})$

<u>ex</u> [Dasbach et al. 2008] Jones poly

Turaev ribbon graph



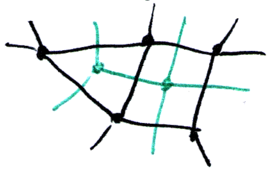
If $n=1$, $m=-A^2-A^{-2}$, $x=A^{-2}$

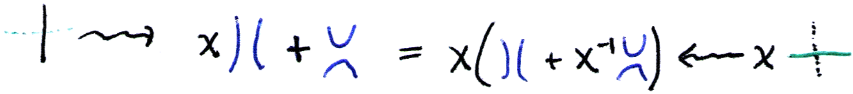


Hence $A^{\#\text{crossings}} BR'_{\partial_K}(A^{-2}, 1, -A^2-A^{-2})$

$= (-A^2-A^{-2})\langle K\rangle_A$.

<u>ex</u> G ribbon graph, has dual G^*





hence

$$BR'_G(x, 1, m) = x^{|E|}\, BR'_{G^*}(x^{-1}, 1, m)$$

* Krushkal polynomial

$G \hookrightarrow \Sigma$ a graph emb. in closed ori. sfc

$$P_{G\hookrightarrow\Sigma}(x,y,a,b) = \sum_{H\subseteq G} x^{b_0(H)-b_0(G)} y^{k(H)} a^{g(H)} b^{g^\perp(H)}$$

with $k(H) = \operatorname{rank}(\ker(H_1(H) \to H_1(\Sigma)))$

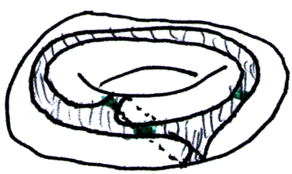
~~theorem~~

Turns out $k(H) = b_0^\perp(H) - b_0(\Sigma)$

<u>def</u> $$P'_{G\hookrightarrow\Sigma}(t,l,w,c) = \sum_{H\subseteq G} t^{|E(H)|} l^{b_0(H)} w^{b_0^\perp(H)} c^{b_0(\partial H)}$$

$$BR'_G(x,n,m) = P'_G(x,n,1,c)$$

Imagine G as a ribbon graph in Σ:



Then 

Let 2DCob be cobordism category of B&W-colored sfc cobordisms.

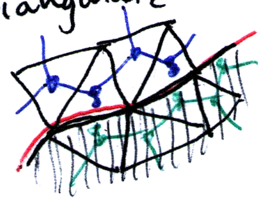
Can define $Z: \text{2DCob} \to \mathbb{C}[l,w,c]$

s.t. $Z((\Sigma,B)) = l^{b_0(B)} w^{b_0(\Sigma - B)} c^{b_0(\partial B)}$

↑ black region

* TQFT

Triangulate :



← use $\bigoplus_{i=1}^{w} \mathbb{C}$ alg

← use $\mathbb{C}^c$ (loops = c)

← use $\bigoplus_{i=1}^{\ell} \mathbb{C}$ alg

More generally, seems part of an extended 2D TQFT of "nonplanar algebras"

* Another approach