Invariants of virtual spatial graphs based on topological graph polynomials

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Overview

The Yamada polynomial [Yamada 1989] is a $U_q(sl(2))$ Reshetikhin-Turaev invariant of spatial graphs.

It has been extended to virtual spatial graphs in a few ways:

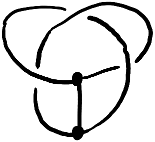
- [Fleming and Mellor 2007]
- [McPhail-Snyder and M. 2018]
- [Deng, Jin, and Kauffman 2018]

Is there a unifying framework to understand these extensions?

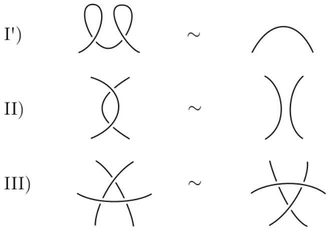
Virtual spatial graphs

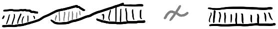
A spatial graph is an embedding of a ribbon graph in S^3.

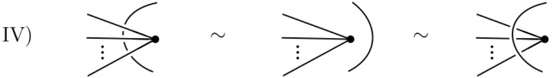
Ex



Like knots & links, they have diagrams up to Reidemeister-like moves



← Regular isotopy to preserve twists: 

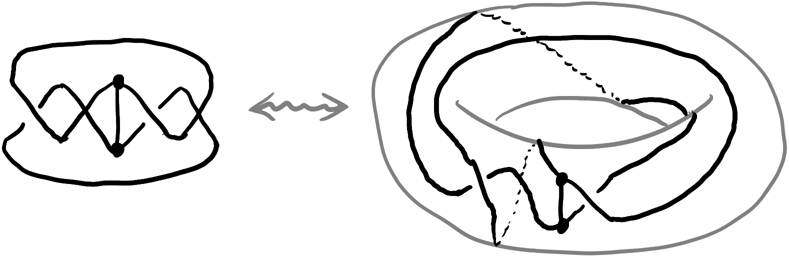


Virtual spatial graphs

A virtual spatial graph is a potentially non-planar spatial graph diagram, modulo the same moves.

"Virtual crossings" are artifacts of non-planarity:

Ex

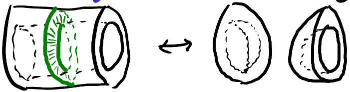


[Kauffman 1999] - Virtual knots

[Fleming and Mellor 2007] - Virtual spatial graphs

Virtual spatial graphs

[Carter, Kamada, Saito 2002] & [Kuperberg 2003]

Thm. Virtual spatial graphs are in one-to-one correspondence with ribbon graphs in thickened closed oriented surfaces modulo stable equivalence (surgery on vertical annuli: ).

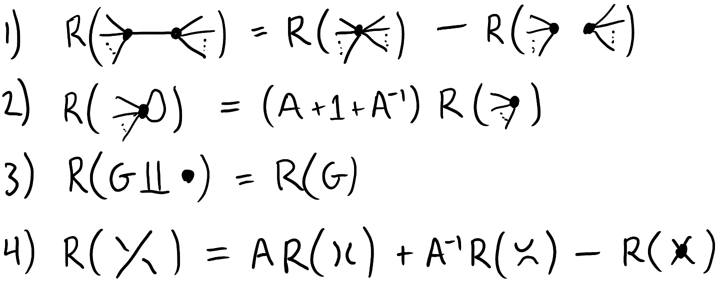
Furthermore, each has a unique representative in the minimal-genus such thickened surface.

Cor. Distinct ("classical") spatial graphs are distinct virtually, too.

The Yamada polynomial [Yamada 1989]

G - spatial graph

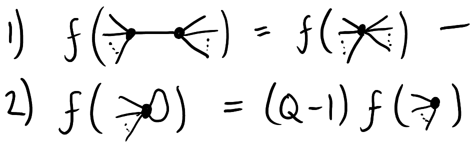
$R(G;A) \in \mathbb{Z}[A^{\pm 1}]$ is Yamada polynomial, defined by



Warning: this is renormalized by $(-1)^{|V|-|E|}$ from the original

Virtual Yamada polynomials

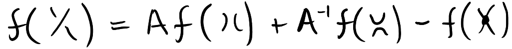
To get invariants of virtual spatial graphs, all we need is a ribbon graph invariant f satisfying



with $Q = (-A^{1/2} - A^{-1/2})^2$

3) $f(G \sqcup \bullet) = f(G)$

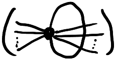
Then: extend by



The flow polynomial

For G a graph and Γ a finite abelian group of order Q, the number of nowhere-zero Γ flows is given by

$$F_G(Q) = \sum_{H \subseteq E(G)} (-1)^{|H|} Q^{b_1(G-H)}.$$

This satisfies the recurrence, with $F($  $) = (Q-1)\,F($  $)$.

[Fleming & Mellor 2007]

Def. Let R^F be the Yamada polynomial based on F.

The "S-polynomial"

[Fendley & Krushkal 2010] observe the flow polynomial of planar graphs can be computed with $TL^{Q^{1/2}}$.

where closed loops evaluate to $Q^{1/2}$.

$$\# := \| - Q^{-1/2} \substack{\cup \\ \cap}$$

2nd Jones-Wenzl projector

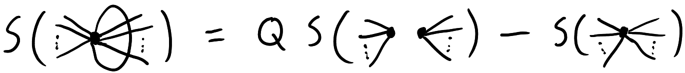
Then normalize by $(Q^{1/2})^{|E|-|V|}$.

The "S-polynomial" [McPhail-Snyder & M. 2018]

Generalizing to non-planar ribbon graphs yields $S_G(Q)$.

$$S_G(Q) = \sum_{H \subseteq E(G)} (-1)^{|H|} Q^{b_1(G-H) - g(G-H)}$$

This satisfies the recurrence, with



Def. Let R^S be the Yamada polynomial based on S.

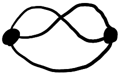
Note For G a link, $R^S_G = R_G$ gives the 2nd colored Jones polynomial.

A criterion for classicality

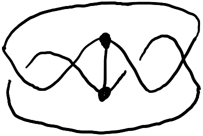
[McPhail-Snyder & M. 2018] (extends [Miyazawa 2006] from virtual links)

Thm. For G a virtual spatial graph,

$R_G^F(A) \neq R_G^S(A) \Rightarrow$ G is not equivalent to a classical spatial graph.

Ex $G =$ 

$$R_G^F = (A + A^{-1})(A + 1 + A^{-1})$$
$$R_G^S = -2(A + 1 + A^{-1})$$

$H =$ 

$$R_H^F = -A^{-2}(A-1)(A+1)^2(A+A^{-1})(A-1+A^{-1})(A+1+A^{-1})$$

$$R_H^S = -A^{-2}(A-1)(A+1)^2(A+1+A^{-1})(A^2-2A+4-2A^{-1}+A^{-2})$$

Interpolation?

[Deng, Jin & Kauffman 2018] define a 2-variable Yamada polynomial for virtual spatial graphs by solving for skein relation coefficients.

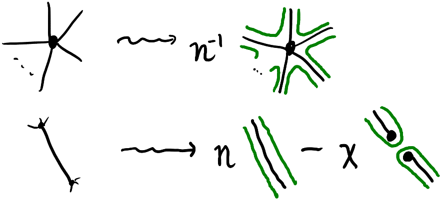
It turns out a renormalization of their polynomial interpolates between R^F and R^S — but how?

The Bollobás–Riordan polynomial [Bollobás, Riordan 2001–2002]

BR_G is a 3-variable ribbon graph invariant generalizing the Tutte polynomial.

[M. unpublished]

The following graphical substitution gives (a version of) $BR_G(n,m,x)$:



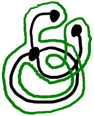
where green loops evaluate to n and black graphs are evaluated according to the $\mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ Frobenius algebra:

1.  $= m$ 

2.  $=$ 

3. $G \amalg \bullet = G$

The Bollobás-Riordan polynomial

Ex $BR($  $)$ $= n^{-1}nn$  $- n^{-1}nx$ 

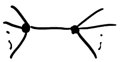
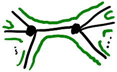
$- n^{-1}xn$  $+ n^{-1}xx$ 

$$= n \cdot nm^2 \qquad - x \cdot n^2 m$$

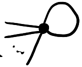
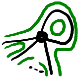
$$- x \cdot n^2 m \qquad + n^{-1} x^2 \cdot n$$

$$= m^2 n^2 - 2mn^2 x + x^2$$

$BR_G(n, m, 1)$

1. $BR\Big($  $\Big) = n^{-1}\Big($  $- \; n^{-1}$  $\Big)$

   $= BR\Big($  $\Big) - BR\Big($  $\Big)$

2. $BR\Big($  $\Big) =$  $- \; n^{-1}$ 

   $= (n^2 m - 1)\, BR\Big($  $\Big)$

3. $BR(\bullet) = n^{-1}$  $= 1$

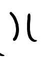
Thus $BR_G(n, m, 1)$ extends to a Yamada polynomial, $Q = n^2 m$.

Generalized Yamada polynomial

For G a virtual spatial graph, $R^{BR}(G; A, n)$ is

1) If G has no crossings, $R^{BR}(G; A, n) = BR_G(n, m, 1)$

where $m = \dfrac{A + 2 + A^{-1}}{n^2}$

2) $R^{BR}\Big($  $\Big) = A\, R^{BR}\Big($  $\Big) + A^{-1} R^{BR}\Big($  $\Big) - R^{BR}\Big($  $\Big)$

Specializations:

- $R^F(G; A) = R^{BR}(G; A, 1)$ $\quad$ ($n = 1$: only black graph)
- $R^S(G; A) = R^{BR}(G; A, -A^{1/2} - A^{-1/2})$ $\quad$ ($m = 1$: only green curves)

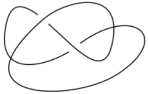
Results and questions

arXiv: 1805.00575

Thm. If G is a classical spatial graph, [M. unpublished]

$$R^{BR}(G; A, n) = R(G; A) \in \mathbb{Z}[A^{\pm 1}]$$

But not sufficient for being classical!



Thm. $R^F_G(-1) = R^S_G(-1) = F_G(0)$. If $\bar{\Omega}(G; \mathbb{Z}/2\mathbb{Z}) = 0$, $R^F_G(1) = F_G(4)$.

Q: There are special local relations at certain (A, n). Do more give R^{BR} relations?

Thm. Each symmetric Frobenius algebra yields a Yamada invariant.
(R^{BR} is from $\mathbb{C}[\mathbb{Z}/m\mathbb{Z}] \otimes \mathrm{End}(\mathbb{C}^n)$.)

Q: Do these invariants come from, say, the Las Vergnas polynomial?

Q: Are there other "R-matrices" beyond ⤫ $= A$)($+ A^{-1}$ ≍ $-$ ✕ ?