3-mfld seminar
Nov 27, 2018
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"What is an alternating knot?"

— Fox acc'd. to Lickorish

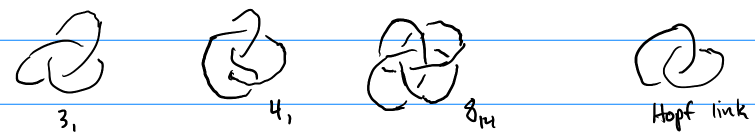
In Nov 2015, Greene & Howie independently answered this question using spanning surfaces.

* Recall: A <u>link</u> is a closed 1-mfld in a 3-mfld (usually S^3) up to isotopy.
(Assume smooth or piecewise linear)

A <u>knot</u> is a one-component link.

For $L \subset S^3$, a <u>diagram</u> is $L \subset S^2 \times [0,1] \subset S^3$ such that the projection of L onto S^2 is an immersion with transverse double pts

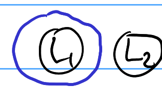
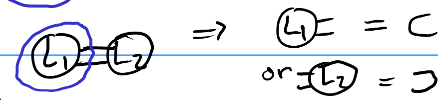
<u>ex</u>



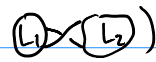
A link is <u>alternating</u> if it has an <u>alternating diagram</u>: crossings alternate over/under around each component.

* Why?

Suppose D is an alternating diagram of a link L.

- L is split iff D is a split diagram 
- L is prime iff D is a prime diagram 

two Tait conjectures: (Thistlethwaite, Kauffman, Murasagi)

- If D is <u>reduced</u> (no ) then $c(D) = c(L)$ (minimal crossing #),
- If D, D' are reduced alt. diags of L, then blackboard framings (writhe) are same.

Greene gives a proof of this

(Menasco)

- If L is prime, non-split, alternating, non-torus link, $S^3 - L$ has hyperbolic geom.

* Checkerboard surfaces

The faces of a 4-regular graph can be 2-colored.

Get B and W surfaces.

$B \cup W$ is almost S^2, except near crossings:

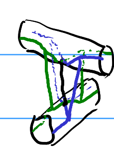
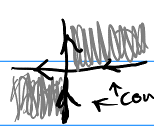


Fig. 1

Can retract onto diagram's S^2, arcs of $B \cap W$ collapse to crossing pts.

Not both are orientable: Sps B oriented. Induce orientation on $L = \partial B$.



so W is not.

$\chi(B) + \chi(W) + \#\text{crossings} = 2$

✻ The characterizations

Thm (Howie, 2015) Let $L \subset S^3$ be a nontrivial nonsplit link, $X = S^3 - \nu(L)$.
L is alternating iff $\exists$ connected spanning surfaces Σ, Σ' s.t.

$$\chi(\Sigma) + \chi(\Sigma') + \underbrace{\tfrac{1}{2} i(\Sigma, \Sigma')}_{c(D)} = 2$$

and • $i(\Sigma, \Sigma') = |\partial(X \cap \Sigma) \cdot \partial(X \cap \Sigma')|$ (in $\partial\nu(L)$)

• or L is a knot

with $i(\Sigma, \Sigma') = |\partial(X \cap \Sigma) \cap \partial(X \cap \Sigma')|$ minimized

Thm (Greene, 2015) Let Y be a $\mathbb{Z}_2HS^3$, $L \subset Y$ a link with $Y - L$ irreducible.
$Y = S^3$ and L is alternating iff $\exists$ spanning sfcs Σ_+, Σ_- that are respectively positive and negative definite with respect to their rational Gordon-Litherland pairings.

We will go over a proof outline of each then spend the talk on understanding Greene's statement.

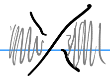
Note: these both give normal surface algorithms to decide if alternating

✻ Outline (Howie)

- Suppose $\partial(X \cap \Sigma) \cap \partial(X \cap \Sigma')$ in minimal position.
- Conditions on ∂ intersections $\Rightarrow$ arcs in $\Sigma \cap \Sigma'$ look like Fig 1.
- Main condition $\Rightarrow \chi(\Sigma \cup \Sigma') = 2 \Rightarrow$ projection is immersed S^2
- When $\Sigma \cap \Sigma'$ has interior loops: By Euler characteristics, one sfc has an innermost disk; do surgery on other sfc. Repeat.
- One component in end is a union of spanning surfaces. Replace Σ, Σ' w/ these
- $\Sigma \cup \Sigma'$ projects to S^2, so get a diagram
- If not alternating, then $\partial(X \cap (\Sigma \cup \Sigma'))$ has a digon, contradicting minimal pos.

* Overview (Greene)

Checkerboard sfcs are definite w/ opp. sign $\Leftrightarrow$ alternating diagram

(If color  then W is $+$-def. and B is $-$-def.)

Given sfcs:

- Can isotope away loops of intersection using definiteness
- Definiteness $\Rightarrow$ arcs of intersection look like Fig. 1.
- So $\Sigma \cup \Sigma'$ retracts onto a sfc in Y with $L \subset \nu(\text{sfc})$
- Form gives # intersection pts in ∂ $\rightsquigarrow$ # arcs

  $\xrightarrow[\text{chars}]{\text{Euler}}$ $\chi(\text{sfc}) = 2$
- Y orientable $\Rightarrow$ sfc has an S^2 comp. S_0
- $Y - L$ irred. $\Rightarrow$ $\partial\nu(S_0)$ bounds disj. balls $\Rightarrow$ $Y = S^3$ and $L \subset \nu(S_0)$
- Get a diagram with Σ, Σ' as checkerboard sfcs $\Longrightarrow$ L alternating.

* Homology 3-spheres

<u>Def</u> For A an ab. gp., a 3-mfld Y is an AHS^3 if $H_i(Y;A) \cong H_i(S^3;A)$ $\forall i$.

<u>ex</u> $Y = L(p,q)$ has
$$H_0(Y) = \mathbb{Z}$$
$$H_1(Y) = \mathbb{Z}_p$$
$$H_2(Y) = 0$$
$$H_3(Y) = \mathbb{Z}$$
so Y is a $\mathbb{Q}HS^3$ and $\mathbb{Z}_r HS^3$ if $\gcd(r,p) = 1$.

$\mathbb{Z}HS^3 \Rightarrow \mathbb{Z}_p HS^3 \Rightarrow \mathbb{Q}HS^3$

$\uparrow$ $H_1(Y)$ has no p-torsion

* Rational linking numbers

For $L_1, L_2 \subset S^3$ disj. ori. links $lk(L_1, L_2) \in \mathbb{Z}$ by summing  $\mapsto +1$  $\mapsto -1$ over crossings between L_1, L_2

Or: $H_1(L_1) \xrightarrow[\cong]{\text{Alex. duality}} H^1(S^3 - \nu(L_1)) \xrightarrow[\cong]{\text{Poin. duality}} H_2(S^3 - \nu(L_1), \partial\nu(L_1))$

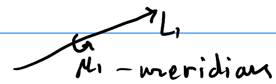
$[L_1] \longmapsto [\Sigma]$ where Σ is a properly emb. sfc.

$\partial\bar{\Sigma} = L$, $\partial\Sigma$ is 0-framed longitude

Then $\operatorname{lk}(L_1,L_2) = [\Sigma]\cdot\underbrace{[L_2]}_{\text{in } H_1(S^3-\nu(L_1))}$

Can do calculations in $\partial\nu(L_1)$. Let $[\lambda_1]=\partial[\Sigma]$ and $[\mu_1]=\partial[D_1]$

s.t. $[\mu_1]\cdot[\lambda_1]=1$ symplectic basis



$([D_1]\cdot[L_1]=1)$

$$[L_2] = \operatorname{lk}(L_1,L_2)\, i_*[\mu_1]$$

Can do a similar construction in Y a $\mathbb{Q}HS^3$, but now Σ is a $\mathbb{Q}$ 2-cycle, and

$\operatorname{lk}(L_1,L_2) = [\Sigma]\cdot[L_2]\in\mathbb{Q}$. Geometrically: Σ wraps around L_1 some number of times, weighted.

Why symmetric?

Let $L=K_1\cup K_2\cup\cdots\subset Y$ oriented, $\mu_i\subset\partial\nu(K_i)$ meridians, $A=Y-\nu(L)$

All gps w/ $\mathbb{Q}$-coeffs: $i:\partial A\hookrightarrow A$

$$\begin{tikzcd}
0=H^2(Y) & H^2(Y,A) \arrow[l] \arrow[d, "\cong\,{[Y]\cap}"'] & H^1(A) \arrow[l, "\delta"'] \arrow[d, "i^*"] & H^1(Y)=0 \arrow[l] \\
 & H_1(\nu(L)) \arrow[d, equal] & H^1(\partial A) \arrow[d, "\cong\,{[\partial A]\cap}"] & \\
 & H_1(L) \arrow[r, dashed, "f"] & H_1(\partial A) &
\end{tikzcd}$$

$f([L_i])=\partial(\text{Alex. dual of } L_i)$

$i_*([\mu_i]\cdot f([L_j]))=\delta_{ij}$. Define $[\lambda_i]\in H_1(\partial\nu(L_i))$ by

$$f([L_i]) = [\lambda_i] + \sum_{k\neq j}\operatorname{lk}(L_i,L_k)[\mu_k]$$

} This really is Seifert-surface-based longitude

$$i_*(f([L_i])\cdot f([L_j])) = \operatorname{lk}(L_i,L_j)-\operatorname{lk}(L_j,L_i)$$
$$\|$$
$$i_*([\partial A]\cap(i^*\alpha_i\cup i^*\alpha_j)) \quad \text{with} \quad [Y]\cap\delta\alpha_k=[L_k]$$
$$=\underbrace{i_*\partial[A]}_{=0}\cap(\alpha_i\cap\alpha_j) = 0.$$

Hence $\operatorname{lk}: H_1(L;\mathbb{Q})\otimes H_1(L;\mathbb{Q})\to\mathbb{Q}$ is symm. bilinear form.

$$\operatorname{lk}(L_1,L_2) = \operatorname{lk}_{L_1\cup L_2}([L_1],[L_2])$$

$\ast$ Gordon – Litherland pairing

Y – $\mathbb{Q}HS^3$

$L \subset Y$ a link

$S \subset Y$ a cpct sfc w/ $\partial S = L$

Y orientable $\Rightarrow$ unit normal bundle $\tilde{N}(S) \subset Y - S$ oriented sfc

Let $p_S : \tilde{N}(S) \to S$ be double cover.

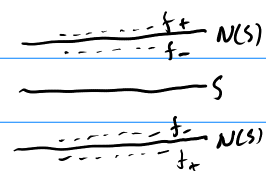
<u>Def</u> $\mathcal{G}_S : H_1(S;\mathbb{Q}) \otimes H_1(S;\mathbb{Q}) \to \mathbb{Q}$

$$[\alpha] \otimes [\beta] \longmapsto \mathrm{lk}(\alpha, p_S^{-1}(\beta))$$ with α, β emb. multicurves

If $Y = S^3$ and S orientable, this is the symmetrized Seifert pairing.

<u>Prop</u> $\mathcal{G}_S$ is symmetric.

<u>Pf</u> Let f_+, f_- denote small pushoffs from $\tilde{N}(S)$.



$$\mathcal{G}_S(\alpha,\beta) = \tfrac{1}{2}\mathrm{lk}(p_S^{-1}\alpha, f_+ p_S^{-1}\beta)$$
$$= \tfrac{1}{2}\mathrm{lk}(p_S^{-1}\beta, f_- p_S^{-1}\alpha) = \mathcal{G}_S(\beta,\alpha). \quad \blacksquare$$

<u>Def</u> S is <u>(+/–)-definite</u> if $\mathcal{G}_S$ is.

<u>Def</u> $\sigma(S)$ = signature of $\mathcal{G}_S$.

If $Y = S^3$ and S a Seifert sfc, $\sigma(S) = \sigma(L)$ (L oriented by ∂S)

Let $K_1, \ldots, K_m$ be comps of L.

<u>Def</u> $e(S) := -\sum_{i=1}^{m} \underbrace{\tfrac{1}{2}\mathcal{G}_S(K_i, K_i)}_{\text{framing induced by } S \text{ on } K_i}$ is <u>Euler number</u>

(in $S^3 \subset B^4$, cap off S with ori. sfc in S^3. is Euler num of normal bundle)

If L oriented, $e(S,L) := -\frac{1}{2} \mathcal{G}_S(L,L)$

$$= e(S) - \sum_{i<j} \mathrm{lk}(K_i, K_j).$$

<u>Thm 2.1</u> (G-L, G) $Y \sim \mathbb{Z}_2 HS^3$, $L \subset Y$ a link, $S \subset Y$ cpct sfc w/ $\partial S = L$.

(1) $\sigma(S) + \frac{1}{2} e(S)$ deps only on L

(if $Y = S^3$, $= \xi(L)$ Murasugi invt $=$ avg of all link sigs)

(2) If L ort., $\sigma(S) + \frac{1}{2} e(S,L)$ deps only on L

(if $Y = S^3$, $= \sigma(L)$)

<u>Prop 3.1</u> If S_1 a definite sfc with $\partial S_1 = L$ and S_2 a sfc w/ $\partial S_2 = L$ and $e(S_1) = e(S_2)$, then $b_1(S_1) \leq b_1(S_2)$. Equality $\Rightarrow$ definite w/ same sign. Thus S_1 is incompressible.

<u>Pf</u> $|\sigma(S_1)| = b_1(S)$, $|\sigma(S_2)| \leq b_1(S)$, $\sigma(S) = \sigma(S')$ by (1). $\blacksquare$

<u>Lemma 3.3</u> If $S' \subset S$, S definite, S' conn. bdry, then S' is definite.

<u>Pf</u> S' semidefinite

$$\begin{array}{ccccc} H_2(S,S') & \longrightarrow & H_1(S') & \xrightarrow{i_*} & H_1(S) \\ \wr\wr & & & & \\ H_2(S/S') = 0 & & & & \end{array}$$

so $\mathcal{G}_{S'}(x,x) = 0$

$\Rightarrow \mathcal{G}_S(i_* x, i_* x) = 0$

$\Rightarrow i_* x = 0 \Rightarrow x = 0.$ $\square$

<u>Lemma 3.4</u> If $X = Y - \nu(L)$ is irred., $S_+, S_- \subset Y$ +/- def sfcs $L = \partial S_+ = \partial S_-$, $(X \cap S_+), (X \cap S_-)$ in minimal pos., then $X \cap S_+ \cap S_-$ has no loops.

$$\frac{1}{2}(e(S_-) - e(S_+)) = \frac{1}{2} i(S_+, S_-)$$