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GRASP
Sep 6, 2018
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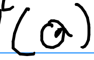
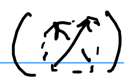
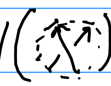
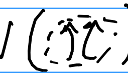
# The Jones Polynomial (part 2)

* Skein relation (Jones 1985)

(similar to Conway 1970 for Alexander 1923, Alex.–Conway poly)

<u>Thm</u> $\exists$ unique $V : \left\{ \begin{matrix} \text{ori. links} \\ \text{in } S^3 \end{matrix} \right\}_{(\text{nonempty})} \longrightarrow \mathbb{Z}[t^{\pm 1/2}]$

s.t. (i) $V($$) = 1$

(ii) $t^{-1} V($$) - t\, V($$) = (t^{1/2} - t^{-1/2})\, V($$)$

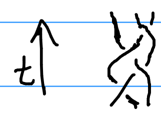
Diagramless! These are links related by modifying <u>tangles</u>, properly embedded compact 1-mflds in $B^3$. (rational 2-tangles)

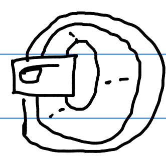
Last time: Kauffman bracket proved existence. (uses diagrams)
Uniqueness: unknottings & $V(L \amalg \bigcirc) = (-t^{1/2} - t^{-1/2}) V(L)$

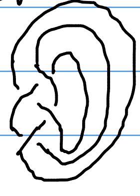
$\Big( \mathbb{Z}[t^{\pm 1/2}][\text{ori. links in 3-mfld } M] / (ii) \rightsquigarrow$ <u>skein modules</u> $\Big)$
$M \cong S^3 \rightsquigarrow \text{module} \cong \mathbb{Z}[t^{\pm 1/2}]$

* Braid group representation (Jones 1985)

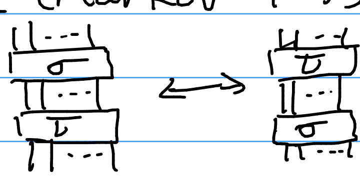
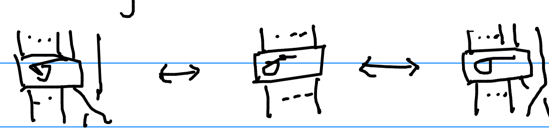
(Artin)
<u>def</u> $B_n$ is $\pi_1$ of configuration space of $n$ distinct pts of $\mathbb{C}$.

<u>ex</u>  $\in B_3$ Gen: $| \cdots \times \cdots |$ , Rels: RII & RIII

<u>def</u> for $\sigma \in B_n$, $\hat{\sigma} =$  is <u>braid closure</u>, a link. (ori.)

<u>ex</u>  (figure-eight knot)

<u>Thm</u> (Alexander 1923) Every (ori.) link is a braid closure.
<u>Thm</u> (Markov 1935) ... up to just

(i)  (ii) 

(2)

(T&L, 1971) (Wenzl 1993 @ roots of unity) $\delta \in R$, $R$ a division ring

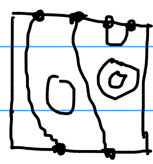
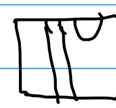
- The Temperley–Lieb category $TL^\delta$ (planar algebra)

objects: for $n \in \mathbb{N}$, $\underline{n} = [0,1]$ with $n$ equally spaced marked points

Morphisms: $TL^\delta(\underline{n}, \underline{m})$ is formal $R$-linear combinations of $\partial$-rel isotopy classes of cobordisms $\underline{n}$ to $\underline{m}$ modulo loops bounding disks $\longleftrightarrow$ mult. by $\delta$

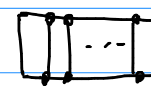
i.e., $C \subset [0,1] \times [0,1]$ a properly embedded 1-mfld s.t. $\partial C = \{0\} \times \text{pts}(\underline{n}) \cup \{1\} \times \text{pts}(\underline{m})$

"planar tangles"

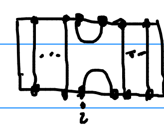
ex  $\in TL^\delta(\underline{2}, \underline{3})$ $= \delta^3$ 

gen by  . $TL^\delta(\underline{0}, \underline{0}) \cong R$.

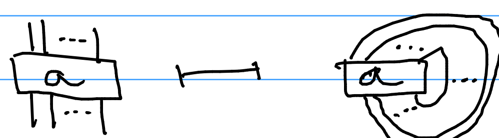
$*: TL^\delta(\underline{n}, \underline{m}) \to TL^\delta(\underline{m}, \underline{n})$ by reflection.

$TL^\delta_n = TL^\delta(\underline{n}, \underline{n})$ is TL algebra. id = 

(a von Neumann alg.)

$\dim TL^\delta_n = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers) $E_i =$ 

$Tr: TL^\delta_n \longrightarrow \mathbb{C}(\delta)$

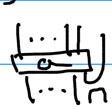


$TL^\delta_n \hookrightarrow TL^\delta_{n+1}$ 

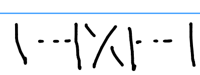
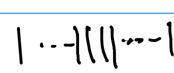
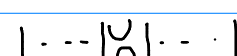
def $tr: \varinjlim_n TL^\delta_n \longrightarrow R$ by $tr(x) = \delta^{-n} Tr(x)$, $x \in TL^\delta_n$

is Markov trace.

(i) $tr(ab) = tr(ba)$ (ii) $tr(id) = 1$

(iii) $tr($  $) = \delta^{-1} tr(a)$

Let $\rho: R[B_n] \longrightarrow TL^\delta_n$, $\delta := -A^2 - A^{-2}$

 $\longmapsto A$  $+ A^{-1}$ 

③

with $\langle \bigcirc \rangle = \delta$

For $\sigma \in B_n$, $\operatorname{tr} \sigma = \delta^{-n} \langle \hat{\sigma} \rangle$, hence

$$V_{\hat{\sigma}}(t) = \delta^{n-1} (-A^{-3})^{w(\sigma)} \operatorname{tr}(\rho(\sigma)) \quad \text{with } A = t^{-1/4}$$

$$w : B_n \to \operatorname{Ab}(B_n) \approx \mathbb{Z} \qquad \text{(writhe)}$$

$$|\cdots|\times|\cdots| \mapsto 1$$

See appendix

(sim. to Jones's def, though his was deformed Burau) "transpose" (?)

(Could also have used $|\cdots|\times|\cdots| \mapsto -A^{-2}\,|\cdots||||\cdots| - A^{-4}\,|\cdots|{}^{\cup}_{\cap}|\cdots|$ to avoid normalization.)

$\ast$ Understanding $TL_n^\delta$

$$\mathbb{C}[S_n] \twoheadrightarrow \operatorname{End}_{sl_2}(V_1^{\otimes n}) \qquad \times = || + {}^{\cup}_{\cap}$$

$$\cap : V_1 \otimes V_1 \to \mathbb{C} \quad = \det$$

$$\bigcirc = -2$$

$$\mathbb{C}[[h]][B_n] \twoheadrightarrow \operatorname{End}_{U_h(sl_2)}(V_1^{\otimes n}) = TL_n^\delta \qquad \delta = -e^h - e^{-h}$$

$$\times = e^{h/2}\,|| + e^{-h/2}\,{}^{\cup}_{\cap}$$

both have a Schur–Weyl duality.

Want to decompose $V_1^{\otimes n}$. ($TL_n^\delta$ generically semisimple) $\delta \neq -2\cos\frac{2\pi k}{n}$

Let $TL_{n,p}$ be (non-unital) subalgebra with $\leq p$ throughstrands ($n+p$ even)

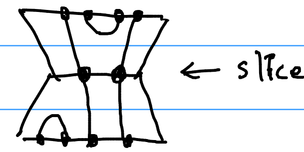
ex $|{}^{\cup}_{\cap} \in TL_{3,1}$

(composition series) filtration: $TL_n = TL_{n,n} \supsetneq TL_{n,n-2} \supsetneq TL_{n,n-4} \supsetneq \cdots$

quotient algebras:

$$Q_{n,p} := TL_{n,p}/TL_{n,p-2} \cong \overset{\text{as left module}}{K_{n,p}} \otimes_R \overset{\text{as right module}}{K^*_{n,p}}$$

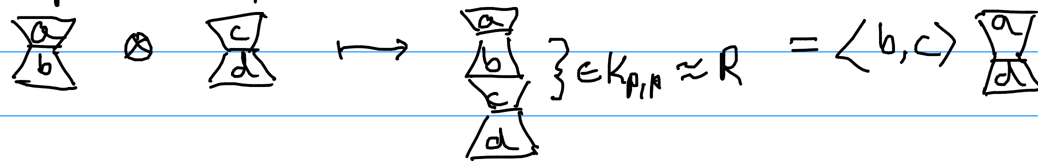
as $TL_n$-bimodule

ex 


where

$K_{n,p} := TL^\delta(p, \underline{n}) / \{\text{fewer than } p \text{ through strands}\}$ is a $TL^\delta_n$-module.

multiplication: $Q_{n,p} \otimes Q_{n,p} \longrightarrow Q_{n,p}$




with $\langle , \rangle : K^*_{n,p} \otimes K_{n,p} \longrightarrow K_{p,p} \cong R$

Consider basis of simple diagrams for $K_{n,n-2r}$. $\{a_i\}_i$

$$\langle a_i^*, a_j \rangle = \begin{cases} \delta^r & \text{if } i=j \\ \delta^{<r} \text{ or } 0 & \text{otherwise} \end{cases}$$

matrix of $\langle , \rangle$ symmetric, and for $|\delta| \gg 0$, diagonal dominates $\Rightarrow \det \neq 0 \Rightarrow \langle , \rangle$ non-degenerate.

Hence $K_{n,p} \otimes K^*_{n,p} \longrightarrow \mathrm{End}_R(K_{n,p})$ is alg. isomorphism,

$a \otimes b^* \longmapsto (c \longmapsto \langle b^*, c\rangle a)$ (so $Q_{n,p}$ simple)

Define $\langle , \rangle : TL^\delta_n \otimes TL^\delta_n \longrightarrow R$ by $\langle a, b\rangle = \mathrm{tr}(ab^*)$.

$\langle ab, c\rangle = \langle b, a^* c\rangle$ diagonal dominance $\Rightarrow$ non-degen. generically

hence $TL^\delta_n$ is generically semisimple.

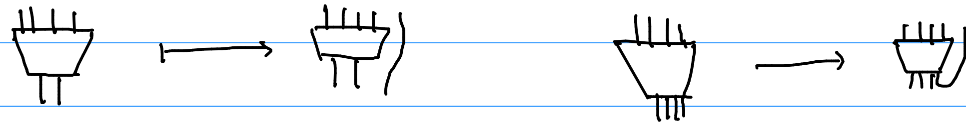
Thus $$TL^\delta_n \cong \bigoplus_{r=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{End}_R(K_{n,n-2r})$$

$\Rightarrow K_{n,n-2r}$ are the simple modules

**Lemma** $\mathrm{Res}_{TL^{\delta}_{n-1}} K_{n,p} \cong K_{n-1,p-1} \oplus K_{n-1,p+1}$

$n,\ n-2r \qquad n-1,\ (n-1)-2r \qquad n-1,\ (n-1)-2(r-1)$

**Pf** $K_{n-1,p-1} \hookrightarrow K_{n,p} \qquad K_{n-1,p+1} \hookrightarrow K_{n,p}$



∎

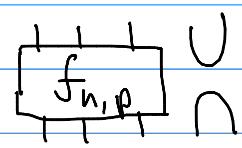
Bratteli diagram

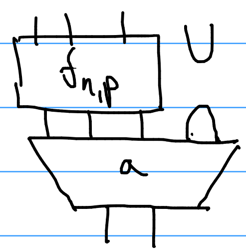
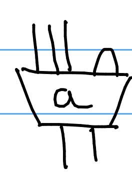
| $n \downarrow$ \ $p \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | | 1 | | | | | |
| 2 | 1 | | 1 | | | | |
| 3 | | 2 | | 1 | | | |
| 4 | 2 | | 3 | | 1 | | |
| 5 | | 5 | | 4 | | 1 | |
| 6 | 5 | | 9 | | 5 | | 1 |

$\dim_k K_{n,p}$ = multiplicity of $V_p$ in $V_1^{\otimes n}$ (Schur-Weyl duality)

$*$ Projectors

semisimple $\Rightarrow$ $\exists\, f_{n,p} \in TL_n$ s.t. $f_{n,p}^2 = f_{n,p}$
and $TL_n f_{n,p} \cong K_{n,p}$. Defined up to right mul. by $TL_n$

**Lemma** $f_{n+2,p} = \delta^{-1}$ 

**Pf** $a \in K_{n+2,l}$  $\neq 0 \Rightarrow$  has exactly $p$ through strands

$\Rightarrow a$ has exactly $p$

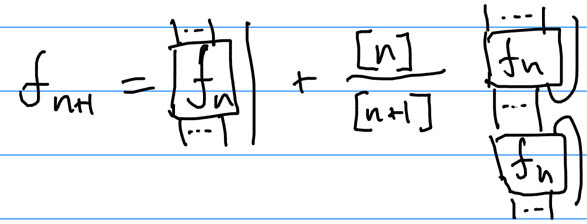
$\Rightarrow l = p$. ∎

⑥

dim $Q_{n,n} = 1 \Rightarrow f_{n,n}$ well-def.

<u>def</u> $f_n = f_{n,n}$ is Jones-Wenzl projector.

Wenzl:



Characterization: (i)  $= 0$

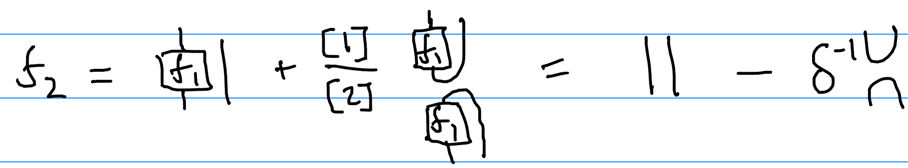
(ii) $f_n^2 = f_n$

$[n] := q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)}$ with $q = A^2$, $\delta = -[2]$

See Morrison's notes.

$f_n : V_1^{\otimes n} \longrightarrow V_1^{\otimes n}$ projects onto $V_n$.

<u>ex</u> $f_1 = |$



for $\delta = -2$, $(sl_2)$

$f_2 = || + \frac{1}{2} \substack{\cup \\ \cap}$

$= \frac{1}{2} || + \frac{1}{2} \left( || + \substack{\cup \\ \cap} \right)$

$= \frac{1}{2} \left( || + X \right)$

so $V_2 \cong \mathrm{Sym}^2 V_1$

Colored Jones polynomial from doing representation to $\mathrm{End}_{U_h(sl_2)}(V_k^{\otimes n})$. Can use $f_n$ to do calc. in $\mathrm{End}_{U_h(sl_2)}(V_1^{\otimes kn})$

Cor $\dim K_{n,n-2r} = \binom{n}{r} - \binom{n}{r-1}$

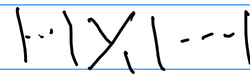
Pf $r=0$ : $\dim K_{n,n} = 1$ $\qquad$ $2r > n$, $\dim = 0$

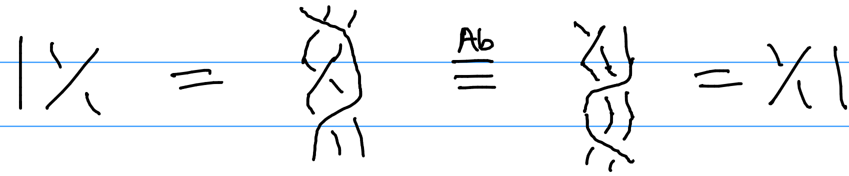
$$\left(\binom{n-1}{r} - \binom{n-1}{r-1}\right) + \left(\binom{n-1}{r-1} - \binom{n-1}{r-2}\right)$$

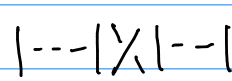
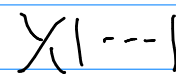
$$= \binom{n}{r} - \binom{n}{r-1} \qquad \square$$

lemma $\displaystyle\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{r} - \binom{n}{r-1}\right)^2 = \frac{1}{n+1}\binom{2n}{n}.$

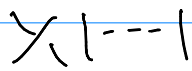
# Appendix

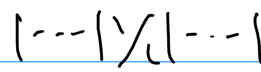
**Thm** $\mathrm{Ab}(B_n) \simeq \mathbb{Z}$ with  $\longmapsto 1$

**Pf** 

hence in $\mathrm{Ab}(B_n)$,  $\equiv$ 

so $\mathbb{Z} \longrightarrow \mathrm{Ab}(B_n)$ is a surjection.

$n \longmapsto$  $^{n}$

$\mathrm{Ab}(B_n) \longrightarrow \mathbb{Z}$ with  $\longmapsto 1$

is a homomorphism. $\blacksquare$