Student 3-mfld seminar

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The sphere Theorem pt 2

* More about ends of groups

Recall: For X a topological space, $\mathcal{E}(X) = \varprojlim_{K \text{ compact}} \pi_0(X-K)$ is space of ends.

(Freudenthal 1931, Hopf 1943)

Thm G a f.g. discrete gp, $G \curvearrowright X$ by covering transformations. If X/G is compact,

(i) $\mathcal{E}(X)$ is ϕ, $\bullet$, $\bullet\bullet$, or Cantor set

(ii) If $G \curvearrowright Y$ with same hypotheses, $\mathcal{E}(Y) \cong \mathcal{E}(X)$.

ex $\mathbb{Z} \curvearrowright \mathbb{R} \to S^1$ so $\mathbb{Z}$ has two ends.

If X is a Cayley graph of f.g. G, then $X/G = \bigvee^n S^1$ is compact.

$\mathcal{E}(G) :\cong \mathcal{E}(X)$.

For X a locally finite CW cplx, let $e(X) = 0, 1, \dots, \infty$, depending on $|\mathcal{E}(X)|$

Prop $e(X) = \sup_K |X-K|$ where K finite subcomplex s.t. each component of $X-K$ is infinite.

Specker 1949: A cohomological approach: Let X be a loc. finite Δ-complex. (1-skeleton is a graph)

Fix $\mathbb{Z}/2\mathbb{Z}$ as coefficient ring.

$C^n(X)$ is all labelings of n-simplices by ring elt.

or: all subsets of set of n-simplices $\left((S \to \mathbb{Z}/2\mathbb{Z}) = 2^S = \mathcal{P}(S) \right)$

For $W \in C^0(X)$, $\delta W = \{\text{edges between } W \text{ and } W^c\}$

Let $C^n_f(X)$ be all finite subsets ("cohomology with compact support")

Let $C^n_e(X) := C^n(X) / C^n_f(X)$.

$[V] \in H^0_e(X)$ has δV a finite collection of edges

and $[V] = [V+F]$ for every $F \in C^0_f(X)$.

Prop $e(X) = \dim H^0_e(X)$.

Pf For $K \subset X$ finite with each component of $X-K$ infinite, let $W_1, \dots, W_n$ be vertex sets of each component. Then $\delta W_i \subset \delta K^{(0)}$ is finite, so $[W_1], \dots, [W_n] \in H^0_e(X)$ and are lin. indep. $\Rightarrow e(X) \leq \dim$.

If $[W_1], \dots, [W_n] \in H^0_e(X)$ are lin. indep, let $K \subset X$ be finite with $\delta W_i \subset K$ and $X-K$ comps infinite. Then W_i constant on each comp. $\Rightarrow e(X) \geq \dim$. ∎

Cor $e(X) = e(X^{(1)})$.

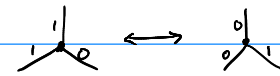
Get a LES $\quad 0 \to H^0_f(X) \to H^0(X) \to H^0_e(X) \to H^1_f(X) \to H^1(X) \to \cdots$

If X is infinite and connected, $H^0_f(X) = 0$ and $H^0(X) = \mathbb{Z}/2\mathbb{Z}$

So: $\quad 0 \to \mathbb{Z}/2\mathbb{Z} \to H^0_e(X) \to \ker(H^1_f(X) \to H^1(X)) \to 0$

$\Rightarrow e(X) \geq 2$ iff ker is nontrivial.

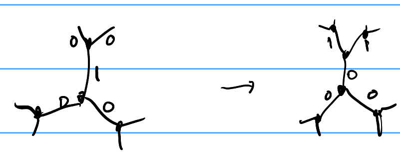
Aside What is ker? $\quad H^1_f(X)$ is finite subsets of edges mod δV for V finite

"Lights-out" game: press vertex,  (toggles incident edges)

generates cohom class.

is in ker if $= \delta W$ for any subset of vertices.

$\Leftarrow$ can turn off all edges in any finite subcomplex of X.

ex  can push "on" edges down subtree

Stallings 1971.

Thm A For G f.g., $e(G) \geq 2$ iff G is a non-triv. amalg. product or HNN extension over a finite subgroup. ("splits over a finite subgroup")

$\leftarrow G = \mathbb{Z}$ is perhaps a special case

Let's prove the Sphere Thm first.

* 3-mflds

Lemma (Kneser) If $S \subset M^3$ is a 2-sided emb. sfc, $\pi_1(S) \to \pi_1(M)$ not injective, then $\exists$ emb. disk $D \subset M^3$ with $D \cap S = \partial D$ and $\underbrace{[\partial D] \text{ nontriv in } \pi_1(S)}_{\text{"essential in } S\text{"}}$.

Pf sketch Let $f: D^2 \to M$ be a nullhomotopy. Use 2-sidedness to push $f(\partial D)$ off S. Transverse $\Rightarrow$ $f^{-1}(S)$ is loops. Take innermost disk. Restriction is either (1) essential boundary, then restrict f to this, or (2) nullhomotopic bdry, so replace with nullhom and push to other side. Then have essential loop in $\partial(M - \nu S)$ that's nullhomotopic in $M - \nu S$: apply loop thm. $\blacksquare$

Prop B Let M^3 compact and $G = \pi_1(M)$ splits over a finite subgp B. Then M has a 2-sided essential emb. sfc S w/ $\pi_1(S) \hookrightarrow B \subset G$.

Pf $G = A *_B C$ or $G = A *_B$. • $K = K(A *_B C, 1)$

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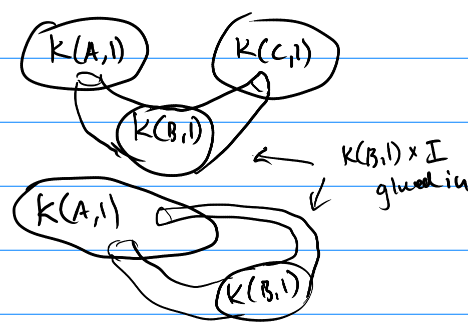


$\mathrm{id}: G \to G$

gives $f: M \to K(G,1)$ up to homotopy.

Put in general position so transverse to $K(B,1)$ (codim-1 in $K(B,1) \times I$)

$f^{-1}(K(B,1))$ is a 2-sided sfc. S.

If $\pi_1(S \to M)$ is not injective, Kneser's lemma gives compression disk D, can homotope f near D to perform compression.

Eventually, π_1-injective. And $\pi_1(S) \subset B$ by construction. $\blacksquare$

(*) If $\pi_1(\mathbb{RP}^2) \longrightarrow \pi_1(M)$ trivial, $(\mathbb{RP}^2 \subset \partial M)$
loop thm $\Rightarrow$ $\exists$ disk, which is 2-sided
since $H_1(D^2; \mathbb{Z}/2\mathbb{Z}) = 0$.
Contradiction since this orients ν of ∂ loop in $\mathbb{RP}^2$!!

Stallings 1971 of Papakyriakopoulos 1957

Thm (Sphere Thm) M^3 compact conn., $\pi_2(M)$ nontriv. Then there is 2-sided S^2 or $\mathbb{RP}^2$ s.t. its $[S^2 \to M] \neq 0$ in $\pi_2(M)$.

Pf If $\pi_1(M)$ finite, $\tilde{M}$ compact, so

$$\begin{aligned} H_0(\tilde{M}) &= \mathbb{Z} \\ H_1(\tilde{M}) &= 0 \\ H_2(\tilde{M}) &= 0 = \pi_2(\tilde{M}) = \pi_2(M) \text{ !!} \\ H_3(\tilde{M}) &= \mathbb{Z} \end{aligned}$$

P.D. ; Hurewicz

Simplify M: If $\mathbb{RP}^2 \subset \partial M$, then (*) (above) $\Rightarrow$ non-triv. in $\pi_2(M)$.

If $S^2 \subset \partial M$, then M would be contractible !! (see (**) (next page))

so non-triv in $\pi_2(M)$

In either case, push-in gives wanted sfc.

If $S \subset \partial M$ not π_1-inj., loop thm gives compressing disk D. ($\partial D \subset S$ essential)

Compress: get $M - \nu(D)$, either (i) disconnected or (ii) connected.

Paste arc in for $\nu(D)$ to get htpy equiv space.

(i) $M \simeq M' \vee M''$ (ii) $M \simeq M' \vee S^1$

So WLOG $\pi_2(M') \neq 0$ and $\pi_2(M') \hookrightarrow \pi_2(M)$ injective.

Eventually, have $M' \subset M$ with
- $\partial M'$ π_1-inj. w/ no S^2 or $\mathbb{RP}^2$
- $\pi_2(M') \hookrightarrow \pi_2(M)$
- $\pi_2(M') \neq 0$.

π_1-inj ∂ $\Rightarrow$ $\partial\widetilde{M'}$ is copies of universal covers of comps of $\partial M'$. ($\mathbb{R}^2$'s)

so $H_1(\partial\widetilde{M'}) = 0$ and $H_2(\partial\widetilde{M'}) = 0$.

$$\pi_2(M') = \pi_2(\widetilde{M'}) \underset{\text{Hurewicz}}{=} H_2(\widetilde{M'}) \underset{\text{LES}}{=} H_2(\widetilde{M'}, \partial\widetilde{M'}) \underset{\text{P.D.}}{=} H^1_f(\widetilde{M'})$$

$$\begin{array}{ccccccccc} 0 \to & H^0_f(\widetilde{M'}) & \to & H^0(\widetilde{M'}) & \to & H^0_e(\widetilde{M'}) & \to H^1_f(\widetilde{M'}) & \to & H^1(\widetilde{M'}) \\ & \| & & \| & & & \# & & \| \\ & 0 & & \mathbb{Z}/2\mathbb{Z} & & & 0 & & 0 \text{ since } \pi_1(\widetilde{M'}) = 0 \\ & \text{since noncompact} & & \text{since connected} & & & \text{by above} & & \end{array}$$

So $e(\widetilde{M'}) \geq 2$. G is deck transform gp of $\widetilde{M'}$, so $e(G) = e(\widetilde{M'})$.

Thm A $\Rightarrow$ $\pi_1(M')$ splits over finite gp.

Prop B $\Rightarrow$ $\exists S \subset M'$ ess. with $\pi_1(S) \hookrightarrow$ finite gp $\Rightarrow$ S is disj. union of 2-sided S^2's and $\mathbb{RP}^2$'s.

$\boxtimes$

Filling in hole from talk:

(**) Stallings writes "It is easy to see that, if ∂M contains a 2-sphere which is contractible in M, then M is itself contractible."

Perhaps this is what he meant: (Recall M compact conn.)

1) If M is nonorientable, $|\pi_1(M)| = \infty$.

Pf For M closed: If $< \infty$, with $\mathbb{Q}$-coeffs, $H_0(M) = \mathbb{Q}$, $H_1(M) = 0$, $H_3(M) = 0$.

$\chi(M) = 1 + b_2$, but $\chi(\text{3-mfld}) = 0$ (use P.D. w/ $\mathbb{Z}/2\mathbb{Z}$ coeffs to see this) so $b_2 < 0$!!

If M not closed, fill ∂ with handlebodies.

2) If $\exists$ nullhomologous $S^2 \subset \partial M$, $\pi_1(M) = 0$.

Pf If not...

Take copies M_1, M_2 of M, and let $M' = M_1 \natural_{S^2} M_2$ (along this S^2 ∂ comp.).

$\pi_1(M') = \pi_1(M_1) * \pi_1(M_2)$ is infinite. Lift S^2 to sphere $\Sigma \subset \widetilde{M'}$, which separates $\widetilde{M'}$ since S^2 separates M', $\widetilde{M'} = N_1 \#_\Sigma N_2$, both noncompact.

$$\underbrace{H_3(N_i, \Sigma)}_{=0} \to H_2(\Sigma) \to H_2(N_i) \qquad \text{so} \quad H_2(\Sigma \to N_i) \text{ is injective.}$$

0 since noncompact (can use excision with $\Sigma = \partial B^3$ for $H_3(N_i;\Sigma)$)

$$\underbrace{H_3(\widetilde{M'})}_{=0} \to H_2(\Sigma) \to H_2(N_1) \oplus H_2(N_2) \to H_2(\widetilde{M'}) \to \underbrace{H_1(\Sigma)}_{=0}$$

$$[\Sigma] \mapsto ([\Sigma], [\Sigma])$$

so $([\Sigma], 0) \mapsto$ nonzero

Hence $H_2(\Sigma \to \widetilde{M'})$ has non-trivial image.

But S^2 is nullhomotopic in $M \subset M'$ hence Σ is in $\widetilde{M'}$ by lifting nullhomotopy.

$\Rightarrow [\Sigma] = 0$ in $H_2(\widetilde{M'})$. !! ∎

3) So M orientable. $\partial[M] = \sum_{\substack{C \subset \partial M \\ \text{a component}}} [C]$. $H_3(M;\partial M) = \mathbb{Z}$ & $[S^2] = 0 \Rightarrow \partial[M] = [S^2]$

so ∂M has only one component.

$$\begin{array}{ccccccc} & & & & H_2(M) & & \\ & & & & \| \text{ P.D.} & & \\ \tilde{H}^0(M) & \to & \tilde{H}^0(\partial M) & \to & H^1(M, \partial M) & \to & \tilde{H}^1(M) \\ \| & & \| & & & & \| \\ 0 & & 0 \text{ (one component)} & & & & 0 \end{array}$$

Hence $H_2(M) = 0$

4) Hurewicz $\Rightarrow \pi_2(M) = 0$. This is enough for a contradiction in ST pf. But,

5) $H_3(M) = 0 \Rightarrow \pi_3(M) = 0$. So by Whitehead Thm, $* \to M$ is htpy equivalence (ie, M contractible)

[end of talk]

* Back to Stallings's thm

$H^0_e(X)$ is a Boolean algebra.

For $[W]$, $[W]^* = [W^c]$ $\quad \delta(W^*) = \delta W$

$[W][U] = [W \cap U]$, $\quad \delta(WU) = (\delta W)U + W(\delta U)$ $\quad$ in path algebra

$ve = \begin{cases} e & \text{if } v = v_1(e) \\ 0 & \text{otherwise} \end{cases}$

$ev = \begin{cases} e & \text{if } v = v_2(e) \\ 0 & \text{otherwise} \end{cases}$

Let $Q(X) = \{ V \in C^0(X) \mid \delta V \text{ is finite} \}$ $\quad$ (represents $H^0_e(X)$)

$V \in Q(X)$ is connected if induced complex for V is connected.

include all edges w/ both endpoints in V.

is nontrivial if $\delta V \notin \delta C^0_f(X)$ $\quad \left(\text{if } \delta V = \delta F \text{ then } \delta[V] = \delta[V+F] = 0 \Rightarrow V+F = 0 \text{ or } 1 \right)$

Let k = width $= \min\limits_{\substack{V \in Q(X) \\ \text{nontrivial}}} |\delta V|$ $\quad$ (well-def. if $e(X) \geq 2$)

nontriv. $V \in Q(X)$ is narrow if $|\delta V| = k$

Prop If V narrow, V is connected.

Pf If not, $V = A \amalg B$ with $\delta A \cap \delta B = \emptyset$. $\quad \delta V = \delta A + \delta B \Rightarrow$ WLOG $A \notin \delta C^0_f(X)$.

Then $|\delta V| > |\delta A|$ since X conn. $\Rightarrow |\delta B| > 0$. !! ∎

Prop If $V_1 \supset V_2 \supset \cdots$ for $V_i \in Q(X)$ narrow, $B = \bigcap\limits_i V_i = V_n$ for some n if nonempty.

Pf For $e \in \partial B$, one end in all V_i, other is outside almost all V_i $\Rightarrow |\partial B| \leq k$.

Hence $\exists n$ s.t. $\delta B \subseteq \delta V_n$ and $B \subseteq V_n$.

$V_n = B \amalg (B + V_n)$, $\quad$ If $B + V_n \neq 0$,

$\delta V_n = \delta B + \delta(B + V_n)$

V_n conn $\Rightarrow \delta B \cap \delta(B + V_n) \neq 0$

$\Rightarrow \delta B + \delta B \cap \delta V_n \neq 0 \Rightarrow \delta B \not\subseteq \delta V_n$. !!

Hence $B = V_n$. ∎

Cor If $v \in V \in Q(X)$ w/ V narrow, $\exists$ minimal such V containing v.

Thm Let $v \in V \in Q(X)$ be narrow & minimal, $V \neq W \in Q(X)$ narrow, then at least one of VW, V^*W, VW^*, V^*W^* is finite.

Pf $\delta(VW) = \delta V\, W + V\delta W$

$\delta(V^*W) = \delta V\, W + V^*\delta W$

$\delta(VW^*) = \delta V\, W^* + V\delta W$

$\delta(V^*W^*) = \delta V\, W^* + V^*\delta W$

$|\delta(VW)| + \cdots + |\delta(V^*W^*)| \leqslant 2|\delta V(W+W^*)| + 2|(V+V^*)\delta W| = 4k$

If, say, $|\delta(VW)| < k$, then VW trivial, so $VW + F = 1$ or 0 for a finite F.

If $VW + F = 1$, $VW \subset V$ so $V^* \subset F$, but V^* is infinite since V nontriv. !!

So $VW = F$ is finite. Then Done.

Else, all narrow. But $v \in VW, VW^* \subsetneq V$. !! ∎

For $[V] \in H^0_e(X)$, let $[V] \cap \mathcal{E}(X)$ denote $\eta \in \mathcal{E}(X)$ s.t. for $K \subset X$ finite with $\delta V \subset K$, $\eta(K) \subset V$ (i.e., V is constant outside K)

If V nontrivial, $[V] \cap \mathcal{E}(X)$, $[V]^* \cap \mathcal{E}(X)$ is a nontrivial partition of clopen sets.

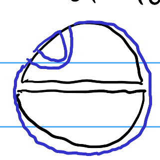
Suppose G a f.g. gp $\curvearrowright X$ a Cayley graph

Let $v \in V \in Q(X)$ be minimal narrow.

$\{gV \mid g \in G\}$ is collection of narrow sets, gV minimal for gv.

Each gives partition of $\mathcal{E}(X)$

Prop $\Rightarrow$ <u>nested</u> partitions. One of four intersections is trivial



Each gV gives edges $g\delta V$ that separate X.

Say e, e' equivalent if $e \in g\delta V$, $e' \in g'\delta V$, and $[gV]$, $[g'V]$ partition $\mathcal{E}(X)$ in same way.

Let T = quotient by this relation. nested $\Rightarrow$ well-def. tree

(and collapse complementary components $\quad$ $G \curvearrowright T$ minimally.

If $e(G) > 2$, if $g\delta V$ and $g'\delta V$ "sufficiently far away" have an end between them (loc. finite)

$\Rightarrow \mathrm{Stab}_G(\text{edge})$ is finite

If $e(G) = 2$, G is virtually $\mathbb{Z}$, so acts on "linear tree".

So, by last time, G is a non-triv. $A *_B C$ or $A *_B$ with B finite.

The converse is to look at some covers of presentation complexes $\blacksquare$