Math 1B: Calculus **Spring 2020**

Discussion 21: Modeling With Differential Equations (9.1)

Instructor: Alexander Paulin Date: Mar 30, 2020

1. Verify that the following differential equations have the given solutions, where y is a function of t.

   (a) For $y' = ay$, the solutions $y = Ce^{at}$ for $C \in \mathbb{R}$.

   $y' = Cae^{at} = a(Ce^{at}) = ay$

   (b) For $y'' + y = 0$, the solutions $y = C\sin(t)$ and $y = C\cos(t)$ for $C \in \mathbb{R}$.

   $y' = C\cos(t)$ | $y' = -C\sin(t)$

   $y'' = -C\sin(t)$ | $y'' = -C\cos(t)$

   $y'' + y = -C\sin(t) + C\sin(t) = 0$ | $y'' + y = -C\cos(t) + C\cos(t) = 0$

   (c) For $ty' = 1$, the solutions $y = \ln(t) + C$ for $C \in \mathbb{R}$.

   $y' = \frac{1}{t} + 0$

   $ty' = t \cdot \frac{1}{t} = 1$

2. Determine the general solution to the given differential equation.[1]

   (a) $y' = 0$.

   $y = \int 0\,dt = C$, so $y = C$ for $C \in \mathbb{R}$

   (b) $y'(t) = \sin(t)$

   $y = \int \sin(t)\,dt = -\cos(t) + C$

   so $y = -\cos(t) + C$

[1] *(Hint: this is integration by another name.)*

(c) $y'(t) = t^2$

$y = \int t^2\,dt = \frac{1}{3}t^3 + C$

(d) $y''(t) = -\frac{1}{t^2}$

$y' = \int -\frac{1}{t^2}\,dt = \frac{1}{t} + C$

$y = \int \left(\frac{1}{t} + C\right) dt = \ln|t| + Ct + D$

so $y(t) = \ln|t| + Ct + D$ for $C, D \in \mathbb{R}$

3. Find the equilibrium solutions[2] to the following differential equations.

$y(t) = C$ means $y'(t) = 0$

(a) $\frac{dy}{dt} = ay$ for $a \neq 0$. (What happens if $a = 0$?)

$\frac{0}{a} = \frac{ay}{a}$

$0 = y$ so $y(t) = 0$ is only equilibrium soln.

(b) $\frac{dy}{dt} = 1 - y^2$.

$0 = 1 - y^2$

$y^2 = 1$ so $y(t) = 1$ and $y(t) = -1$

$y = \pm 1$

(c) $\frac{dy}{dt} = 1 + y^2$.

$0 = 1 + y^2$

$-1 = y^2$ so no equilibrium solutions

(d) $\frac{dy}{dt} = ay(b - y)$ for $a \neq 0$.

$0 = ay(b-y)$ so $y(t) = 0$ and $y(t) = b$ are equilibrium solns

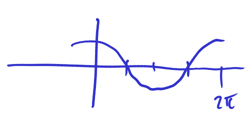
$0 = y(b-y)$

$y = 0$ or $b - y = 0$

$b = y$

(e) $\frac{dy}{dt} = \cos(y)$.

$0 = \cos(y)$



so $y(t) = \pi(n + \frac{1}{2})$ for $n \in \mathbb{Z}$

$y = \frac{\pi}{2} + \pi n$ for $n \in \mathbb{Z}$

(many equilibrium solutions!)

[2] An *equilibrium solution* is a constant-valued solution $y(t) = c$ for some $c \in \mathbb{R}$.

4. An object of mass m hanging from an ideal spring with spring constant k obeys the second-order differential equation

$$\frac{d^2x}{dt^2} = -\tfrac{k}{m}x,$$

where $x(t)$ is the vertical displacement through time.

(a) Show that $x(t) = A\cos(t\sqrt{k/m}) + B\sin(t\sqrt{k/m})$ is a solution, where $A, B \in \mathbb{R}$.

$$x'(t) = -\sqrt{\tfrac{k}{m}}\, A\sin\left(t\sqrt{\tfrac{k}{m}}\right) + \sqrt{\tfrac{k}{m}}\, B\cos\left(t\sqrt{\tfrac{k}{m}}\right)$$

$$x''(t) = -\tfrac{k}{m} A\cos\left(t\sqrt{\tfrac{k}{m}}\right) - \tfrac{k}{m} B\sin\left(t\sqrt{\tfrac{k}{m}}\right)$$

$$= -\tfrac{k}{m}\left(A\cos\left(t\sqrt{\tfrac{k}{m}}\right) + B\sin\left(t\sqrt{\tfrac{k}{m}}\right)\right)$$

$$= -\tfrac{k}{m}\, x(t)$$

(b) Solve the initial value problem where $x(0) = 1$ and $x'(0) = 0$, representing releasing the object from one unit of displacement at rest.

$$1 = x(0) = A + 0$$

$$0 = x'(0) = 0 + \sqrt{\tfrac{k}{m}}\, B$$

so $A = 1$ and $B = 0$

$$x(t) = \cos\left(t\sqrt{\tfrac{k}{m}}\right)$$

(c) Solve the initial value problem where $x(0) = 0$ and $x'(0) = 1$, representing giving the object a unit-sized kick at time zero.

$$0 = x(0) = 0 + B$$

$$1 = x'(0) = \sqrt{\tfrac{k}{m}}\, A + 0$$

so $B = 0$ and $A = \sqrt{\tfrac{m}{k}}$

$$x(t) = \sqrt{\tfrac{m}{k}}\, \sin\left(t\sqrt{\tfrac{k}{m}}\right)$$

5. Suppose $g(t)$ is a function. Write the general solution $y(t)$ for $y'(t) = g(t)$ using a definite integral, given the initial condition $y(0) = C$.

$$y(t) = \int_0^t g(s)\,ds + C$$

Check: $y'(t) = \frac{d}{dt}\int_0^t g(s)\,ds + 0$

$$= g(t)$$

6. Guess the general solution $y(t)$ for $y^{(n)} = 0$. *(Hint: What kinds of functions become zero after repeated differentiation?)*

- Saw $y' = 0$ has $y = C$ as soln.
- $y'' = 0$, $y' = \int 0\,dt = C$, $y = \int C\,dt = Ct + D$
- $y''' = 0$, $y'' = \int 0\,dt = C$, $y' = \int C\,dt = Ct + D$, $y = \int (Ct+D)\,dt$ $= \frac{1}{2}Ct^2 + Dt + E$
- Guess: $y(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_{n-1}t^{n-1}$ is general soln to $y^{(n)} = 0$.

7. Suppose $y = f(t)$ and $y = g(t)$ are two solutions to $y'' + ay' + by = 0$. Show that $y = Af(t) + Bg(t)$ is also a solution, for all constants $A, B \in \mathbb{R}$.

$$y' = Af' + Bg'$$
$$y'' = Af'' + Bg''$$

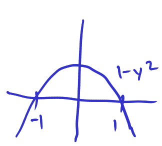
$$\begin{aligned} y'' + ay' + by &= (Af'' + Bg'') + a(Af' + Bg') + b(Af + Bg) \\ &= A(f'' + af' + bf) + B(g'' + ag' + bg) \\ &= A \cdot 0 + B \cdot 0 \\ &= 0 \end{aligned}$$

8. An equilibrium solution $y(t) = c$ for a first-order differential equation $y'(t) = g(y)$ is called *stable* if there is an interval (a, b) containing c such that $g(y)$ is positive for $y \in (a, c)$ and $g(y)$ is negative for $y \in (c, b)$. Analyze the stability of equilibrium solutions in problem 3. Can you make intuitive sense of this terminology?

will be more clear in 9.2

(a) $y' = ay$ $y = 0$ stable if $a < 0$

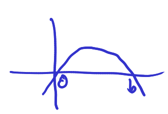
(b) $y' = 1 - y^2$



$y = -1$ is not stable
$y = 1$ is stable

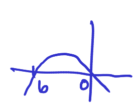
(c) $y' = 1 + y^2$ (no equilib. solns)

(d) $y' = ay(b-y)$ if $a > 0$ and $b > 0$,



$y = b$ stable

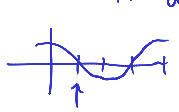
if $a > 0$ and $b < 0$,



$y = 0$ stable

if $a < 0$ and $b > 0$, $y = 0$ stable
if $a < 0$ and $b < 0$, $y = b$ stable

(e) $y' = \cos(y)$



$y = \pi(2n + \frac{1}{2})$ stable, $n \in \mathbb{Z}$