Models for Population Growth : Solutions

1

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

(1) ——— $$\underbrace{\frac{dP}{P} \cdot \frac{1}{1 - P/M}}_{\text{isolate}} = k\,dt$$

isolate

$$\frac{dP}{P}\,\frac{1}{(1 - P/M)}$$

$$= \frac{M\,dP}{P(M-P)}$$

PFD

$$M = \frac{A}{P} + \frac{B}{M-P}$$

$$M = A(M-P) + BP$$

$$B = 1, \quad A = 1$$

$$\int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP \qquad \begin{aligned} u &= M-P \\ du &= -dP \end{aligned}$$

$$= \ln|P| + (-\ln|M-P|) + C_1$$

$$= \ln|P| - \ln|M-P| + C_1 \quad \text{———} \quad (2)$$

Plug (2) into (1)

$$\ln|P| - \ln|M-P| = kt - C_1$$

$$e^{\ln\left|\frac{M-P}{P}\right|} = e^{-kt + C_1}$$

$$\left|\frac{M-P}{P}\right| = e^{C_1} \cdot e^{-kt}$$

$$\frac{M-P}{P} = \pm e^{C_1} \cdot e^{-kt} \quad , A = \pm e^{C_1}$$

$$\frac{M}{P} - 1 = Ae^{-kt}$$

$$\frac{M}{P} = Ae^{-kt} + 1$$

$$\frac{1}{P} = \frac{Ae^{-kt} + 1}{M}$$

$$\boxed{P = \frac{M}{Ae^{-kt} + 1}}$$ (solution to Logistic Diff EQ)

$$P(0) = \frac{M}{A + 1}$$

$$1 + A = \frac{M}{P_0}$$

$$A = \frac{M}{P_0} - 1$$

$$\boxed{A = \frac{M - P_0}{P_0}}$$ A given initial condition $P(0) = P_0$

2

Rewrite

a) $\frac{dP}{dt} = 0.4P\left(1 - \frac{1}{400}P\right)$, $\boxed{M=400}$ carrying capacity

b) $\frac{dP}{dt}(0) = 0.4(50) - 0.001(50)^2$

$\boxed{= 17.5}$

c) $P = \frac{M}{1+Ae^{-kt}} = \frac{400}{1+Ae^{-0.4t}}$, $k = 0.4$

$A = \frac{400-50}{50} = 7$

$= \frac{400}{1+7e^{-0.4t}}$

$P(t) = 200 = \frac{400}{1+7e^{-0.4t}}$

solve for t

$1+7e^{-0.4t} = 2$

$e^{-0.4t} = \frac{1}{7}$

$\ln e^{-0.4t} = \ln\frac{1}{7}$

$t = \frac{\ln\frac{1}{7}}{-0.4}$

$\boxed{t = 4.86}$

3) Given:

- In 2000, P = 6.1 bil
- Birth rate = 35 - 40 mil / yr
- Death rate = 15 - 20 mil / yr
- M = 20 bil

a)

$$\text{growth rate} = \text{average birth rate} - \text{average death rate}$$

$$k = 37.5 - 17.5$$

$$k = 20 \text{ mil / yr} = 0.02 \text{ bil / yr}$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

$$\boxed{\frac{dP}{dt} = 0.02\, P\left(1 - \frac{P}{20}\right)}$$

3 b)

First we need to find $P(t)$

General solution for logistic DE:

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{20}{1 + Ae^{-0.02t}}$$

In 2000, $P(0) = 6.1 = \frac{20}{1 + A} \Rightarrow A = \frac{20}{6.1} - 1$

$$A = 2.3$$

$$\Rightarrow P(t) = \frac{20}{1 + 2.3e^{-0.02t}}$$

$\Rightarrow$ In 2010, $P(10) = \frac{20}{1 + 2.3e^{-0.02 \cdot 10}} = \boxed{6.74 \text{ bil}}$

c) In ~~20~~ 2100, $\boxed{P(100) = 15.25 \text{ bil}}$

In 2500, $\boxed{P(500) = 20 \text{ bil}}$

4)

Given:

- At $t = 0$, $P(0) = 400$
- $M = 10000$
- $P(1) = 3 \times P(0) = 1200$

a) General sol'n for logistic DE:

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

$$P(0) = \frac{10000}{1 + A} = 400 \Rightarrow A = \frac{10000}{400} - 1 = 24$$

$$P(1) = \frac{10000}{1 + 24e^{-k}} = 1200 \Rightarrow 24e^{-k} = \frac{10000}{1200} - 1$$

$$24e^{-k} = \frac{22}{3}$$

$$-k = \ln\left(\frac{22}{3 \times 24}\right)$$

$$k = 1.19$$

$$\Rightarrow \boxed{P(t) = \frac{10000}{1 + 24e^{-1.19t}}}$$

4b)

$$P(t) = \frac{10000}{1+24e^{-1.19t}} = 5000$$

$$\Rightarrow \quad 1 + 24e^{-1.19t} = \frac{10000}{5000} = 2$$

$$e^{-1.19t} = \frac{1}{24}$$

$$t = -\frac{1}{1.19} \cdot \ln\left(\frac{1}{24}\right)$$

$$\boxed{t = 2.67 \text{ years}}$$