Fibonacci sequence:

n	0	1	2	3	4	5	6
f_n	1	1	2	3	5	8	13

(linear recurrence relation)

combinatorics

exponential generating fn:

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{5x^4}{4!} + \frac{8x^5}{5!} + \cdots$$

$$F'(x) = \sum_{n=1}^{\infty} f_n \frac{x^{n-1}}{(n-1)!} = 0 + \frac{1}{0!} + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{5x^3}{3!} + \frac{8x^4}{4!} + \cdots$$

$$F''(x) = \sum_{n=2}^{\infty} f_n \frac{x^{n-2}}{(n-2)!} = 0 + 0 + \frac{2}{0!} + \frac{3x}{1!} + \frac{5x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

$$F'' = F' + F$$

$$F'' - F' - F = 0$$

Initial value problem

$F(0) = 1$

$F'(0) = 1$

$$r^2 - r - r = 0$$

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$$

$$-\varphi^{-1} = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots$$

$$F(x) = c_1 e^{\varphi x} + c_2 e^{-\varphi^{-1} x} \qquad F(0) = c_1 + c_2$$

$$F'(x) = c_1 \varphi e^{\varphi x} - c_2 \varphi^{-1} e^{-\varphi^{-1} x} \qquad F'(0) = c_1 \varphi - c_2 \varphi^{-1}$$

$$\begin{cases} 1 = c_1 + c_2 \\ 1 = c_1 \varphi - c_2 \varphi^{-1} \end{cases} \quad \times \varphi$$

$$\varphi = c_1 \varphi^2 - c_2$$

$$\varphi + 1 = c_1(\varphi^2 + 1) \rightsquigarrow c_1 = \frac{\varphi + 1}{\varphi^2 + 1} =$$

$$c_1 = \frac{5+\sqrt{5}}{10} \qquad c_2 = \frac{5-\sqrt{5}}{10}$$

$$F(x) = \frac{5+\sqrt{5}}{10} e^{\varphi x} + \frac{5-\sqrt{5}}{10} e^{-\varphi^{-1} x}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{5+\sqrt{5}}{10} \varphi^n + \frac{5-\sqrt{5}}{10} (-\varphi^{-1})^n \right)}_{f_n} \frac{x^n}{n!}$$

$$n \gg 0, \quad f_n \approx \frac{5+\sqrt{5}}{10} \varphi^n$$

2.1. $y'' - xy' - y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y(0) = c_0$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y'(0) = c_1$$

$$\left(y^{(n)}(0) = n!\, c_n\right)$$

$$0 = y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$(m = n-2)$

$$= \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$(n = m)$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) c_{n+2} - (n+1) c_n\right) x^n$$

$0 = (n+2)(n+1) c_{n+2} - (n+1) c_n$ for $n \geq 0$

$$c_{n+2} = \frac{\cancel{(n+1)} c_n}{(n+2)\cancel{(n+1)}} \xrightarrow{(m = n+2)} c_m = \frac{c_{m-2}}{m} \text{ for } m \geq 2$$

k	n	c_n
0	0	$c_0 = 1$
	1	$c_1 = 0$
1	2	$c_2 = \frac{c_0}{2} = \frac{1}{2}$
	3	$c_3 = \frac{c_1}{3} = 0$
2	4	$c_4 = \frac{c_2}{4} = \frac{1}{4\cdot 2}$
	5	$c_5 = 0$
3	6	$c_6 = \frac{c_4}{6} = \frac{1}{6\cdot 4\cdot 2}$
	7	$c_7 = 0$
4	8	$c_8 = \frac{c_6}{8} = \frac{1}{8\cdot 6\cdot 4\cdot 2}$

Guess: $c_{2k+1} = 0 \qquad k \geq 0$

$$c_{2k} = \frac{1}{2\cdot 4\cdot 6\cdot 8\cdots(2k)} = \frac{1}{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot(2\cdot 4)\cdots(2\cdot k)} = \frac{1}{2^k\cdot 1\cdot 2\cdot 3\cdot 4\cdots k} = \frac{1}{2^k k!}$$

$$y = \sum_{k=0}^{\infty} c_{2k} x^{2k}$$

$$\boxed{y = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}}$$

$$= \sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!}$$

$$\boxed{y = e^{\frac{1}{2}x^2}}$$

2.3 $\quad y'' + x^2 y' + xy = 0 \qquad y(0) = 0 = c_0 \qquad y'(0) = 1 = c_1$

$$0 = y'' + x^2 y' + xy$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$(n-2 = m+1)$

$$= \sum_{m=-1}^{\infty} (m+3)(m+2) c_{m+3} x^{m+1} + \sum_{n=0}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$(n=m)$

$(n=-1)$

$$= 2c_2 + \sum_{n=0}^{\infty} \left((n+3)(n+2) c_{n+3} + (n+1) c_n\right) x^{n+1}$$

$$\begin{cases} 0 = 2c_2 \\ 0 = (n+3)(n+2) c_{n+3} + (n+1) c_n & \text{for } n \geq 0 \end{cases}$$

$$c_{n+3} = \frac{-(n+1) c_n}{(n+3)(n+2)} \xrightarrow{m = n+3} c_m = \frac{(m-2) c_{m-3}}{m(m-1)} \quad \text{for } m \geq 3$$

k	n	c_n
	0	$c_0 = 0$
0	1	$c_1 = 1$
	2	$c_2 = 0$
	3	$c_3 = \frac{c_0}{3\cdot 2} = 0$
1	4	$c_4 = \frac{2c_1}{4\cdot 3} = \frac{2}{4\cdot 3}$
	5	$c_5 = \frac{3\cdot c_2}{5\cdot 4} = 0$
2	7	$c_7 = \frac{5\cdot c_4}{7\cdot 6} = \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}$
3	10	$c_{10} = \frac{8\cdot c_7}{10\cdot 9} = \frac{8\cdot 5\cdot 2}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$

$c_{3k} = 0 \quad$ for $k \geq 0$

$c_{3k+2} = 0$

$$c_{3k+1} = \frac{2\cdot 5\cdot 8 \cdots (3k-1)}{(3\cdot 4)(6\cdot 7)(9\cdot 10)\cdots((3k)\cdot(3k+1))}$$

$$\boxed{y = \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1}}$$