Fibonacci:

n	0	1	2	3	4	5	
f_n	1	1	2	3	5	8	...

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \frac{1}{0!} + \frac{1x}{1!} + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{5x^4}{4!} + \frac{8x^5}{5!} + \cdots$$

$$F'(x) = \sum_{n=1}^{\infty} f_n \frac{x^{n-1}}{(n-1)!} = \frac{1}{0!} + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{5x^3}{3!} + \frac{8x^4}{4!} + \cdots$$

$$F''(x) = \sum_{n=2}^{\infty} f_n \frac{x^{n-2}}{(n-2)!} = \frac{2}{0!} + \frac{3x}{1!} + \frac{5x^2}{2!} + \frac{8x^3}{3!}$$

$$F'' = F' + F$$

F solves $y'' - y' - y = 0$

initial values $y(0) = 1$, $y'(0) = 1$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\varphi = \frac{1+\sqrt{5}}{2}$ $\quad -\varphi^{-1} = \frac{1-\sqrt{5}}{2}$

$$y = C_1 e^{\varphi t} + C_2 e^{-\varphi^{-1} t} \rightsquigarrow y(0) = C_1 + C_2$$

$$y' = C_1 \varphi e^{\varphi t} - C_2 \varphi^{-1} e^{-\varphi^{-1} t} \rightsquigarrow y'(0) = C_1\varphi - C_2\varphi^{-1}$$

For F:

$1 = y(0) = C_1 + C_2$

$1 = y'(0) = C_1\varphi - C_2\varphi^{-1}$

$C_2 = 1 - C_1$

$1 = C_1\varphi - (1 - C_1)\varphi^{-1}$

$\varphi = C_1\varphi^2 - (1 - C_1)$ $\qquad \varphi^2 = \varphi + 1$

$\varphi = C_1(\varphi + 1) - 1 + C_1$

$\varphi = \varphi C_1 - 1$

$\varphi + 1 = \varphi C_1$

$C_1 = \frac{\varphi + 1}{\varphi} = \frac{\varphi^2}{\varphi} = \varphi$

$C_2 = 1 - C_1 = 1 - \varphi = -\varphi^{-1}$

so $F(x) = \varphi e^{\varphi t} - \varphi^{-1} e^{-\varphi^{-1} t}$

$$= \sum_{n=0}^{\infty} \left(\varphi \cdot \frac{(\varphi t)^n}{n!} - \varphi^{-1} \frac{(-\varphi^{-1} t)^n}{n!} \right)$$

$$\boxed{f_n = \varphi^{n+1} + (-\varphi^{-1})^{n+1}}$$

(maybe...)

2.1. $y'' - xy' - y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = c_0 \qquad\quad = c_1$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + \cdots$$

$$0 = y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$(m = n-2)$

$$= \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$(n = m)$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) c_{n+2} - n c_n - c_n\right) x^n$$

$$0 = (n+2)(n+1) c_{n+2} - (n+1) c_n \quad \text{for } n \geq 0$$

$$c_{n+2} = \frac{c_n}{n+2}$$

$(m = n+2)$

$$c_m = \frac{c_{m-2}}{m} \quad \text{for } m \geq 2$$

k	n	c_n
0	0	$c_0 = 1$
	1	$c_1 = 0$
1	2	$c_2 = \frac{c_0}{2} = \frac{1}{2}$
	3	$c_3 = \frac{c_1}{3} = 0$
2	4	$c_4 = \frac{c_2}{4} = \frac{1}{4 \cdot 2}$
	5	$c_5 = \frac{c_3}{5} = 0$
3	6	$c_6 = \frac{c_4}{6} = \frac{1}{6 \cdot 4 \cdot 2}$

Guess:

$c_{2k+1} = 0 \quad$ for $k \geq 0$

$$c_{2k} = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2k)} = \frac{1}{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot \cdots \cdot (2\cdot k)} = \frac{1}{2^k \cdot 1 \cdot 2 \cdot 3 \cdot \cdots \cdot k} = \frac{1}{2^k\, k!}$$

$$y = \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}x^2\right)^k}{k!} = \boxed{e^{\frac{1}{2}x^2}}$$

$$y' = x e^{\frac{1}{2}x^2}$$

2.2. $y'' + x^2 y = 0 \qquad y(0) = 1 \quad y'(0) = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$1 = y(0) = c_0$$
$$0 = y'(0) = c_1$$

$$0 = y'' + x^2 y = \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{(m = n-2)} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{(m = n+2)}$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m + \sum_{m=2}^{\infty} c_{m-2} x^m$$

$$= \underbrace{2c_2}_{(m=0)} + \underbrace{6 c_3 x}_{(m=1)} + \sum_{m=2}^{\infty} \left((m+2)(m+1) c_{m+2} + c_{m-2} \right) x^m$$

$$\begin{cases} 0 = 2c_2 \\ 0 = 6c_3 \\ 0 = (m+2)(m+1) c_{m+2} + c_{m-2} \quad \text{for } m \geq 2 \end{cases}$$

$$c_{m+2} = \frac{-c_{m-2}}{(m+2)(m+1)} \qquad (n = m+2) \qquad c_n = \frac{-c_{n-4}}{n(n-1)}$$

k	n	c_n
0	0	$c_0 = 1$
	1	$c_1 = 0$
	2	$c_2 = 0$
	3	$c_3 = 0$
1	4	$c_4 = \frac{-c_0}{4 \cdot 3} = \frac{-1}{4 \cdot 3}$
	5	0
	6	0
	7	0
2	8	$c_8 = \frac{-c_4}{8 \cdot 7} = \frac{1}{8 \cdot 7 \cdot 4 \cdot 3}$
3	12	$c_{12} = \frac{-c_8}{12 \cdot 11} = \frac{-1}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$
4	16	$c_{16} = \frac{-c_{12}}{16 \cdot 15} = \frac{1}{16 \cdot 15 \cdot 12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$

$k \geq 0 \quad c_{4k+1} = c_{4k+2} = c_{4k+3} = 0$

$$c_{4k} = \frac{(-1)^k}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12 \cdot \ldots (4k-1)(4k)}$$

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12 \cdots (4k-1)(4k)}$$

$$4 \cdot 8 \cdot 12 \cdot \ldots \cdot 4k = 4^k k!$$

$$c_{4k} = \frac{(-1)^k}{4^k k! \cdot 3 \cdot 7 \cdot 11 \cdots (4k-1)}$$