$(*)$ $ay'' + by' + cy = G(t)$

$\downarrow$ complementary equation

$(**)$ $ay'' + by' + cy = 0$

$\downarrow$ auxiliary equation

$a\lambda^2 + b\lambda + c = 0$

find roots $\lambda = r_1, r_2$

Case I $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$

Soln to $(**)$ is $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Case II $r_1 = r_2 = r$

Soln to $(**)$ is $y = C_1 e^{rt} + C_2 t e^{rt}$
$= (C_1 + C_2 t) e^{rt}$

Case III $r_1, r_2 = \alpha \pm \beta i$

Soln to $(**)$ is $y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

Now for $(*)$, get a particular solution.

Case A $G(t) = (b_0 + b_1 t + \cdots + b_n t^n) e^{kt}$

Ansatz: $y_p = (d_0 + d_1 t + \cdots + d_n t^n) e^{kt}$ if $k \neq r_1, r_2$

$y_p = t(d_0 + d_1 t + \cdots + d_n t^n) e^{kt}$ if $k = r_1$ or $k = r_2$

Case B $G(t) = (b_0 + b_1 t + \cdots + b_n t^n) \cos(kt) e^{dt}$

Ansatz: $y_p = (d_0 + d_1 t + \cdots + d_n t^n) \cos(kt) e^{dt}$
$+ (e_0 + d_1 t + \cdots + e_n t^n) \sin(kt) e^{dt}$

But, if $d \pm ki = r_1, r_2$, multiply by $t$

Plug ansatz into $ay'' + by' + cy = G(t)$, solve for
$d_0, d_1, \ldots, e_0, e_1, \ldots$ $\rightsquigarrow$ get $y_p$

Soln to $(*)$:
$y = y_p +$ homog. soln

ex $9y'' + y = e^{2x}$

$$9\lambda^2 + 1 = 0$$

$$\lambda^2 = \frac{-1}{9}$$

$$\lambda = \pm\frac{1}{3}i$$

$$y_h = C_1 \cos\left(\tfrac{1}{3}x\right) + C_2 \sin\left(\tfrac{1}{3}x\right)$$

ansatz: $y_p = Ae^{2x}$

$$y_p' = 2Ae^{2x}$$

$$y_p'' = 4Ae^{2x}$$

$$9(4Ae^{2x}) + (Ae^{2x}) = e^{2x}$$

$$37Ae^{2x} = e^{2x}$$

$$37A = 1$$

$$A = \frac{1}{37} \rightsquigarrow y_p = \frac{1}{37}e^{2x}$$

$$y = \frac{1}{37}e^{2x} + C_1\cos\left(\tfrac{1}{3}x\right) + C_2\sin\left(\tfrac{1}{3}x\right)$$

$$y'' - 4y' + 4y = x - \sin(x) \quad \begin{cases} (*) \; y'' - 4y' + 4y = x \\ (**) \; y'' - 4y' + 4y = -\sin(x) \end{cases}$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2 \qquad y_h = (C_1 + C_2 t)e^{2t}$$

$(*)$ $\quad x = (0 + x)e^{0t}$

ansatz $\quad y_p = (A + Bx)e^{0t} = A + Bx$

$$y_p' = B$$

$$y_p'' = 0$$

$$(0) - 4(B) + 4(A + Bx) = 4A - 4B + 4Bx = x$$

$$\begin{cases} 4A - 4B = 0 \\ 4B = 1 \end{cases} \rightsquigarrow B = \frac{1}{4} = A$$

$(**)$ $\quad -\sin(x) = -e^{0x}\sin(x)$

$$0 \pm 1i$$

ansatz $\quad y_p = A\cos(x) + B\sin(x)$

$$y_p' = -A\sin(x) + B\cos(x)$$

$$y_p'' = -A\cos(x) - B\sin(x)$$

$$(-A\cos(x) - B\sin(x)) - 4(-A\sin(x) + B\cos(x)) + 4(A\cos(x) + B\sin(x))$$

$$= \cos(x)(-A - 4B + 4A) + \sin(x)(-B - 4A + 4B) \qquad = -\sin(x)$$

$$\begin{cases} 3A - 4B = 0 \\ -4A + 3B = -1 \end{cases} \longrightarrow \begin{cases} 3A - 4B = 0 \\ -A - B = -1 \end{cases} \longrightarrow \begin{cases} 0 - 7B = -3 \\ -A - B = -1 \end{cases}$$

$$B = 3/7$$

$$A = 1 - B = \frac{4}{7}$$

$$y_p = \frac{4}{7}\cos(x) + \frac{3}{7}\sin(x)$$

$y_1$ solves $(*)$

$y_2$ solves $(**)$

$$(y_1 + y_2)'' - 4(y_1 + y_2)' + 4(y_1 + y_2)$$

$$= y_1'' + y_2'' - 4y_1' - 4y_2' + 4y_1 + 4y_2$$

$$= (y_1'' - 4y_1' + 4y_1) + (y_2'' - 4y_2' + 4y_2)$$

$$= x + -\sin x$$

so $\quad y = y_1 + y_2$ solves orig. eqn.

All together: $\quad y = \frac{1}{4} + \frac{1}{4}x + \frac{4}{7}\cos(x) + \frac{3}{7}\sin(x) + (C_1 + C_2 t)e^{2t}$

ex
$$y'' + 2y' + 10y = \underbrace{x^2 e^{-x} \cos(3x)}_{-1 \pm 3i}$$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 10}}{2}$$

$$= -1 \pm 3i$$

ansatz $y_p = x\left((a_0 + a_1 x + a_2 x^2)\, e^{-x}\cos(3x) + (b_0 + b_1 x + b_2 x^2) e^{-x} \sin(3x)\right)$

ex
$$y'' - 6y' + 9y = e^{3x} + \underbrace{x \sin(2x)\, e^{0x}}_{0 \pm 2i}$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

$$\lambda = 3, 3$$

$$y_p = a x^2 e^{3x} + (b_0 + b_1 x)\cos(2x) + (c_0 + c_1 x)\sin(2x)$$