Non-homogeneous linear diff. eqs with constant coeffs. (undetermined coefficients)

ex

$$y' - 2y = 0$$
$$r - 2 = 0$$
$$r = 2$$
$$y = Ce^{2t}$$

ex

$$y' \underbrace{-2}_{P=-2} y = \underbrace{e^{4t}}_{Q=e^{4t}}$$

$$I = e^{\int P\,dt} = e^{-2t}$$

$$y = \frac{1}{I}\int IQ\,dt = e^{2t}\int e^{2t}\,dt$$
$$= e^{2t}\left(\tfrac{1}{2}e^{2t} + C\right)$$
$$y = \tfrac{1}{2}e^{4t} + Ce^{2t}$$

ex

$$y' \overbrace{-2}^{P} y = \overbrace{e^{2t}}^{Q}$$

$r-2=0$ "r=2"

$$r = 2$$

$$y_p = Ate^{2t} \cancel{+ Be^{t}}$$
$$y_p' = A(e^{2t} + 2te^{2t})$$
$$= A(1+2t)e^{2t}$$

$$y_p' - 2y_p = A(1+2t)e^{2t} - 2Ate^{2t}$$
$$= Ae^{2t}$$

$$Ae^{2t} = e^{2t} \Rightarrow A = 1$$

so: $y = te^{2t} + Ce^{2t}$

ex

$$y' - 2y = e^{4t}$$

$r-2=0$ "r=4"

$r=2$

$$y_p = Ae^{4t}$$
$$y_p' = 4Ae^{4t}$$

$$y_p' - 2y_p = 4Ae^{4t} - 2\cdot Ae^{4t}$$
$$= 2Ae^{4t}$$

$2Ae^{4t} = e^{4t}$, so $A = \frac{1}{2}$

$$y = \tfrac{1}{2}e^{4t} + Ce^{2t}$$

$$\boxed{\begin{aligned} I &= e^{-2t} \\ y &= \tfrac{1}{I}\int IQ\,dt = e^{2t}\int 1\,dt \\ &= e^{2t}(t + C) = te^{2t} + Ce^{2t} \end{aligned}}$$

ex

$$y' - 2y = \cos(t)$$

$r=2$ $\qquad \underbrace{r = \pm i}$

$$y_p = A\cos(t) + B\sin(t)$$

$$y_p' = -A\sin(t) + B\cos(t)$$

$$y_p' - 2y_p = (B-2A)\cos(t) + (-A-2B)\sin(t)$$

$$\begin{cases} 1 = B - 2A \\ 0 = -A - 2B \end{cases}$$

$A = -2B$

$\to 1 = B - 2(-2B) = B + 4B = 5B$

$\Rightarrow B = \frac{1}{5}$

$A = -2B = -2\left(\frac{1}{5}\right) = -\frac{2}{5}$

$$\boxed{y = -\frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) + Ce^{2t}}$$

$\cos(t) = e^{0t}\cos(1t)$

$0 \pm 1i$

$\hookrightarrow A\cos(t) + B\sin(t)$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

$$\frac{ax+b}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$r = 1, 2$

$\frac{1}{x-r} \xleftrightarrow{\text{analogous}} e^{rx}$

ex $y'' - 3y' + 2y = e^{4t}$ "$r=4$"

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r = 1, 2$

$y_p = Ae^{4t}$

$y_p' = 4Ae^{4t}$

$y_p'' = 16Ae^{4t}$

$y_p'' - 3y_p' + 2y_p = 16Ae^{4t} - 3 \cdot 4Ae^{4t} + 2 \cdot Ae^{4t}$

$= 6Ae^{4t} = e^{4t}$

$A = \frac{1}{6}$

$y = \frac{1}{6}e^{4t} + c_1 e^t + c_2 e^{2t}$

ex $y'' - 3y' + 2y = e^{2t} + e^{3t}$

$r = 1, 2$ "$r = 2, 3$"

$y_p = Ate^{2t} + Be^{3t}$

$y_p' = A(e^{2t} + 2te^{2t}) + 3Be^{3t}$

$y_p'' = A(2e^{2t} + 2e^{2t} + 4te^{2t}) + 9Be^{3t}$

$= A(4+4t)e^{2t} + 9Be^{3t}$

$y_p'' - 3y_p' + 2y_p = Ae^{2t}\left((4+4t) - 3(1+2t) + 2 \cdot t\right)$

$+ Be^{3t}(9 - 3 \cdot 3 + 2 \cdot 1)$

$= Ae^{2t}(1) + Be^{3t}(2) = e^{2t} + e^{3t}$

$\begin{cases} A = 1 \\ 2B = 1 \end{cases}$

$y = te^{2t} + \frac{1}{2}e^{3t} + c_1 e^t + c_2 e^{2t}$

ex $y'' + 1 = \cos(t)$

$r^2 + 1 = 0$ $r = \pm i$

$r = \pm i$ $y_p = At\cos(t) + Bt\sin(t) \ldots$