Homogeneous second-order linear differential equations with constant coefficients

$$ay'' + by' + cy = 0 \qquad a, b, c \text{ constants}$$

1) write auxiliary/characteristic eqn

$$y^{(n)} \longmapsto \lambda$$

$$a\lambda^2 + b\lambda + c = 0$$

2) solve for λ, roots λ_1, λ_2 (with multiplicity)

3a) real roots, $\lambda_1 \neq \lambda_2$

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

3b) $\lambda = \lambda_1 = \lambda_2$

$$y = (C_1 + C_2 t) e^{\lambda t} = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$

3c) non-real roots ($b^2 - 4ac < 0$)

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a}$$

$$= A \pm iB$$

$$y = (C_1 \cos(Bt) + C_2 \sin(Bt)) e^{At}$$

$$= C_1 \cos(Bt) e^{At} + C_2 \sin(Bt) e^{At}$$

$$\left(= C \cos(Bt + \varphi) e^{At}\right)$$

C, φ are the constants

<u>ex</u>

$$y'' - 3y' + 2y = 0 \qquad y(0) = 1,\ y'(0) = -1$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 1, 2$$

$$y = c_1 e^t + c_2 e^{2t}$$

$$y' = c_1 e^t + 2c_2 e^{2t}$$

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ -1 = y'(0) = c_1 + 2c_2 \end{cases}$$

$$c_1 = 1 - c_2$$

$$-1 = (1 - c_2) + 2c_2 = 1 + c_2$$

$$c_2 = -2$$

$$c_1 = 3$$

$$\boxed{y(t) = 3e^t - 2e^{2t}}$$

$$0 = y'' - 3y' + 2y = \frac{d^2}{dt^2}y - 3\frac{d}{dt}y + 2y$$

$$= \left(\frac{d^2}{dt^2} - 3\frac{d}{dt} + 2\right)y$$

$$0 = \left(\frac{d}{dt} - 2\right)\underbrace{\left(\frac{d}{dt} - 1\right)y}_{f}$$

$$\begin{cases} 0 = \left(\frac{d}{dt} - 2\right)f = f' - 2f \\ f = \left(\frac{d}{dt} - 1\right)y = y' - y \end{cases}$$

$$f' = 2f \xrightarrow{\text{separation}} f = Ae^{2t}$$

$$\underbrace{y' - y}_{P=-1} = \underbrace{Ae^{2t}}_{Q} \qquad I = e^{\int P\,dt} = e^{-t}$$

$$y = \frac{1}{I}\int IQ\,dt = e^t \int Ae^t\,dt = e^t(Ae^t + C)$$

$$= Ae^{2t} + Ce^t$$

ex $y'' - 2y' + y = 0$ $y(0) = 1$ $y'(0) = -1$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1, 1$$

$$y = (c_1 + c_2 t) e^t$$

$$y' = (c_1 + c_2 t) e^t + c_2 e^t$$

$$\begin{cases} 1 = y(0) = c_1 \\ -1 = y'(0) = c_1 + c_2 \end{cases}$$

$$c_2 = -2$$

$$\boxed{y = (1 - 2t) e^t}$$

$$0 = y'' - 2y' + y$$

$$= \left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\right) y$$

$$0 = \left(\frac{d}{dt} - 1\right) \underbrace{\left(\frac{d}{dt} - 1\right) y}_{f}$$

$$\begin{cases} 0 = \left(\frac{d}{dt} - 1\right) f = f' - f \\ f = \left(\frac{d}{dt} - 1\right) y = y' - y \end{cases}$$

$$f' = f \rightsquigarrow f = Ae^t$$

$$\underset{P=-1}{y' - y} = \underbrace{Ae^t}_{Q} \qquad I = e^{\int P dt} = e^{-t}$$

$$y = \frac{1}{I} \int IQ\, dt = e^t \int A\, dt = e^t (At + c)$$

ex $y'' + y' + y = 0 \qquad y(0)=1 \quad y'(0)=-1$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot 1}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$

$y = \left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) e^{-\frac{1}{2}t}$

$y' = \left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)\left(\frac{-1}{2}\right) e^{-\frac{1}{2}t}$

$+ \left(-\frac{\sqrt{3}}{2} c_1 \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2} c_2 \cos\left(\frac{\sqrt{3}}{2}t\right)\right) e^{-\frac{1}{2}t}$

$$\begin{cases} 1 = y(0) = c_1 \\ -1 = y'(0) = c_1\left(\frac{-1}{2}\right) + \frac{\sqrt{3}}{2} c_2 \end{cases}$$

$-\frac{1}{2} = \frac{\sqrt{3}}{2} c_2$

$\boxed{c_2 = \frac{-1}{\sqrt{3}} \quad c_1 = 1}$

ex $y'' = 0 \qquad y(0) = 0 \qquad y(1) = 3$

$\lambda^2 = 0$

$\lambda = 0, 0$

$y = (c_1 + c_2 t)e^{0t} = c_1 + c_2 t$

$$\begin{cases} 0 = y(0) = c_1 \\ 3 = y(1) = c_1 + c_2 \end{cases}$$

$c_1 = 0, \ c_2 = 3$

$\boxed{y = 3t}$