Homogeneous linear second-order differential equations with constant coefficents

$$ay'' + by' + cy = 0$$ for a, b, c constants
y is a function of t

1) Write down auxiliary/characteristic eqn

$$y^{(n)} \longmapsto \lambda^n$$

$$a\lambda^2 + b\lambda + c = 0$$

2) Find the roots $\lambda_1, \lambda_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3) Write soln. down, depending on discriminant, in a way that gives easily-real solns

3a) $b^2 - 4ac > 0$, so λ_1, λ_2 are real and $\lambda_1 \neq \lambda_2$

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

3b) $b^2 - 4ac = 0$, $\lambda = \lambda_1 = \lambda_2$ has multiplicity two

$$y = (C_1 + C_2 t) e^{\lambda t}$$
$$= C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$

3c) $b^2 - 4ac < 0$, so $\lambda_1, \lambda_2 = \dfrac{-b}{2a} \pm \dfrac{i\sqrt{4ac - b^2}}{2a}$
$$= A \pm Bi$$

$$Y = (C_1 \cos(Bt) + C_2 \sin(Bt)) e^{At}$$
$$\left(= D_1 e^{\lambda_1 t} + D_2 e^{\lambda_2 t} = D_1 e^{(A+Bi)t} + D_2 e^{(A-Bi)t} \right)$$

ex $y'' - 3y' + y = 0$ $\quad y(0)=1,\ y'(0) = -1$

$\lambda^2 - 3\lambda + 1 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 1, 2$

$y = C_1 e^t + C_2 e^{2t}$

$y' = C_1 e^t + 2C_2 e^{2t}$

$1 = y(0) = C_1 + C_2$

$-\ \ -1 = y'(0) = C_1 + 2C_2$

$2 = 0 - C_2$

$C_2 = 2$

$C_1 = -1$

$$\boxed{y = -e^t + 2e^{2t}}$$

ex $0 = y'' - 3y' + y = \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + y$

$= \left(\frac{d^2}{dt^2} - 3\frac{d}{dt} + 1\right) y$

$0 = \left(\frac{d}{dt} - 2\right)\underbrace{\left(\frac{d}{dt} - 1\right) y}_{f}$

$$\begin{cases} 0 = \left(\frac{d}{dt} - 2\right) f = f' - 2f \\ f = \left(\frac{d}{dt} - 1\right) y = y' - y \end{cases}$$

$f' = 2f \quad \underset{\text{separation}}{\rightsquigarrow} \quad f = Ae^{2t}$

$y' - y = \underbrace{Ae^{2t}}_{Q}$ $\quad P = -1$

$I = e^{\int P\,dt} = e^{\int -1\,dt} = e^{-t}$

$y = \frac{1}{I}\int IQ\,dt = e^t\left(\int e^{-t} A e^{2t}\,dt\right)$

$= e^t A \int e^t\,dt = e^t A(e^t + B)$

$$\boxed{y = Ae^{2t} + ABe^t}$$

ex $\quad y'' - 2y' + y = 0 \qquad y(0) = 1 \quad y'(0) = -1$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1, 1$$

$$y = (c_1 + c_2 t)e^t \qquad 1 = y(0) = c_1$$

$$y' = (c_1 + c_2 t)e^t + c_2 e^t \qquad -1 = y'(0) = c_1 + c_2$$

$$c_1 = 1, \quad c_2 = -2$$

$$\boxed{y = (1 - 2t)e^t}$$

$$0 = y'' - 2y' + y$$

$$= \left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\right) y = \left(\frac{d}{dt} - 1\right)\underbrace{\left(\frac{d}{dt} - 1\right) y}_{f}$$

$$0 = \left(\frac{d}{dt} - 1\right) f = f' - f$$

$$f = \left(\frac{d}{dt} - 1\right) y = y' - y$$

$$f' = f, \quad f = Ae^t$$

$$y' \underbrace{-}_{P=-1} y = \underbrace{Ae^t}_{Q} \qquad I = e^{\int P dt} = e^{-t}$$

$$y = \frac{1}{I}\int IQ\,dt = e^t \int e^{-t} A e^t\, dt = e^t A \int dt$$

$$= e^t A (t + C) = (AC + At)e^t$$

ex $y'' + y' + y = 0 \qquad y(0) = 1 \quad y'(0) = 0$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$y = \left(C_1 \cos\left(\tfrac{\sqrt{3}}{2} t\right) + C_2 \sin\left(\tfrac{\sqrt{3}}{2} t\right)\right) e^{-t/2}$$

$$y' = \left(C_1 \cos\left(\tfrac{\sqrt{3}}{2} t\right) + C_2 \sin\left(\tfrac{\sqrt{3}}{2} t\right)\right)\left(\tfrac{-1}{2}\right) e^{-t/2}$$

$$+ \left(-C_1 \tfrac{\sqrt{3}}{2} \sin\left(\tfrac{\sqrt{3}}{2} t\right) + C_2 \tfrac{\sqrt{3}}{2} \cos\left(\tfrac{\sqrt{3}}{2} t\right)\right) e^{-t/2}$$

$$1 = y(0) = C_1$$

$$0 = y'(0) = C_1\left(\tfrac{-1}{2}\right) + C_2 \tfrac{\sqrt{3}}{2}$$

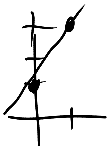
$$C_1 = 1$$

$$C_2 = \frac{1}{\sqrt{3}}$$

$$\boxed{y = \left(\cos\left(\tfrac{\sqrt{3}}{2} t\right) + \tfrac{1}{\sqrt{3}} \sin\left(\tfrac{\sqrt{3}}{2} t\right)\right) e^{-t/2}}$$

ex Boundary value problem

$y'' = 0 \qquad y(0) = 1 \quad y(1) = 3$



$\lambda^2 = 0$

$\lambda = 0, 0$

$y = (c_1 + c_2 t)e^{0t} = c_1 + c_2 t$

$1 = y(0) = c_1 \qquad c_1 = 1$

$3 = y(1) = c_1 + c_2 \qquad c_2 = 2$

$$\boxed{y = 1 + 2t}$$