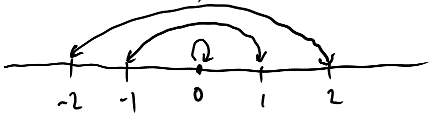
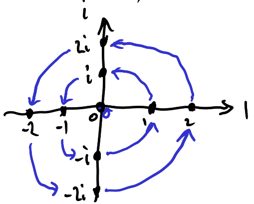
Complex numbers

- multiplication by -1 for $\mathbb{R}$



- multiplication by i for $\mathbb{C}$ $\quad i^2 = -1$



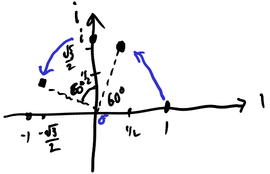
is rotation 90° counterclockwise

- $i^2 = -1$

mult. by i^2, rotate 90° CCW twice
$\rightarrow$ rotation by 180°

hence mult. by -1 is rotation by 180° !

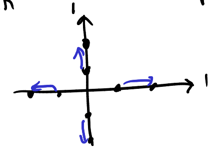
- mult by $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ $\qquad \theta = \frac{\pi}{3}$ (60°)



$$e^{i\theta} i = \cos(\theta) i + i \sin(\theta) i$$
$$= -\sin(\theta) + i\cos(\theta)$$

it's rotation by θ counterclockwise

- mult. by $r \in \mathbb{R}$ $\qquad r = 2$

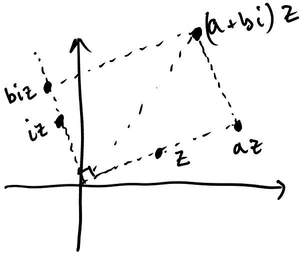


scale by r

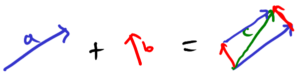
- mult by $re^{i\theta}$

is rotation by θ CCW
with simultaneous scaling by r

- mult by $a+bi$
    1) take scaling by factor of a
    2) take scaled-by-b rotation by $90°$ CCW
    3) add these together



$(a+bi)\,z$



$a + b = c$

Aside: A complex number is a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

(a matrix with orthogonal columns)

$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

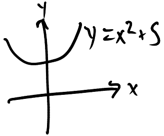
$a+bi = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$e^{i\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$re^{i\theta} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$

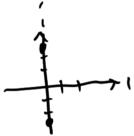
Polynomial roots

ex $x^2+5=0$



$x^2=-5$

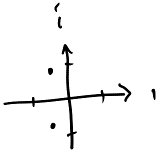
$x=\pm\sqrt{-5} = \pm\sqrt{-1}\sqrt{5} = \pm i\sqrt{5}$



ex $x^2+x+1=0$

disc. $=1^2-4\cdot 1\cdot 1=-3<0$

$x=\dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2\cdot 1} = \dfrac{-1\pm\sqrt{-3}}{2} = \dfrac{-1\pm\sqrt{3}\,i}{2}$



ex $x^2=1$ $\quad x=re^{i\theta}$

$r^2e^{2i\theta}=1=1\cdot e^{0i}$

$e^{2i\theta}=e^{0i}$

$2i\theta = 0i + 2\pi i n$

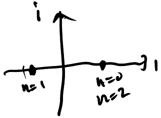
$n\in\mathbb{Z}$

$r^2=1 \rightsquigarrow r=1$

$2\theta = 2\pi n$

$\theta = \pi n$

$x=e^{\pi i n}$ $\quad n\in\mathbb{Z}$

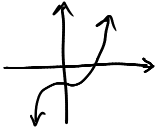


$x=e^0, e^{\pi i}$

$x=1,-1$

ex $x^3-1=0$

$=(x-1)(x^2+x+1)$

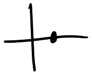


$x=re^{i\theta}$

$(re^{i\theta})^3-1=0$

$r^3e^{3i\theta}-1=0$

$r^3e^{3i\theta}=1$



$1=e^{i\alpha}$ $\quad \alpha=2\pi n$

$n\in\mathbb{Z}$

$1=e^{2\pi n i}$

$r^3e^{3i\theta}=e^{2\pi n i}$

$r^3=1$ and $e^{3i\theta}=e^{2\pi n i}$

$r=1$

$3i\theta = 2\pi n i$

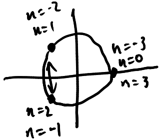
$3\theta = 2\pi n$

$\theta = \dfrac{2\pi n}{3}$

$x=e^{2\pi i n/3}$, $n\in\mathbb{Z}$

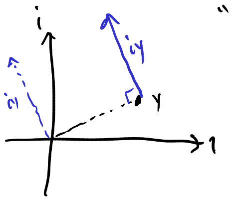
$n\in\{0,1,-1\}$

$x=1, e^{2\pi i/3}, e^{-2\pi i/3}$

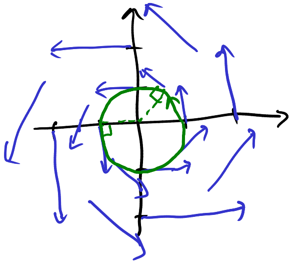


$y = e^{it}$ $\qquad \frac{d}{dt} y = \frac{d}{dt} e^{it} = i e^{it} = iy$

hence y is a soln to $y' = iy$



"deriv. is orthogonal to the position"



$\left(\frac{d}{dt} + i\right) y = 0$

$y' + iy = 0$

$y' = -iy$

$y = Ae^{-it}$

$\left(\frac{d}{dt} - i\right) y = 0$

$y' - iy = 0$

$y' = iy$

$y = Be^{it}$

The point: it makes sense that e^{it} traces a circle

$x^2 + 1 = (x+i)(x-i)$

ex

$y'' + y = 0$

$= \frac{d^2 y}{dt^2} + y = \left(\frac{d^2}{dt^2} + 1\right) y = \left(\left(\frac{d}{dt}\right)^2 + 1\right) y = \left(\frac{d}{dt} + i\right)\left(\frac{d}{dt} - i\right) y = 0$

$y = \left\{\text{solns to } \left(\frac{d}{dt} + i\right) y = 0\right\} + \left\{\text{solns to } \left(\frac{d}{dt} - i\right) y = 0\right\}$

$$y = Ae^{-it} + Be^{it}$$

$$y = A(\cos(-t) + i\sin(-t)) + B(\cos(t) + i\sin(t))$$

$$= (A+B)\cos(t) + (-A+B)\, i\sin(t)$$

$$y = C\cos(t) + D\sin(t)$$