The sum of convergent and divergent series

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Theorem 8 in section 11.2 says (among other things) that if both \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge, then so do \( \sum_{n=1}^{\infty} (a_n + b_n) \) and \( \sum_{n=1}^{\infty} (a_n - b_n) \).

Today I gave the example of a difference of divergent series which converges (for instance, when \( a_n = b_n \)), but I misspoke about what Theorem 8 says about the sum of a convergent and divergent series: the result is in fact divergent.

We will show that if the sum is convergent, and one of the summands is convergent, then the other summand must be convergent. Suppose \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} (a_n + b_n) \) converge. Then \( \sum_{n=1}^{\infty} (a_n + b_n) - \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n \) converges, by Theorem 8. In particular, if \( \sum_{n=1}^{\infty} a_n \) converges and \( \sum_{n=1}^{\infty} b_n \) diverges, so must \( \sum_{n=1}^{\infty} (a_n + b_n) \).

Another way of saying this is by the contrapositive: if the difference of two series diverges, then one of the two series must diverge. Use \( \sum_{n=1}^{\infty} (a_n + b_n) \) and \( \sum_{n=1}^{\infty} a_n \) along with the fact that \( \sum_{n=1}^{\infty} a_n \) is known to converge.