Polynomials are continuous functions

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In this note, we will prove from first principles that polynomials are continuous functions. It is meant to give the general flavor of ε-δ proofs for the general calculus student. For almost all students, limits are their first introduction to formal mathematics, and they are a fairly difficult first concept to absorb. Always remember: drops wear down the stone, not by strength but by constant falling.

By an *open interval*, we mean a set of numbers (a, b), with $a < b$, defined to be all $x \in \mathbb{R}$ such that $a < x < b$. An example of an open interval is $(-1, 1)$, which is all numbers x such that $-1 < x < 1$ (i.e., an interval without the end points). We may also let a or b be $\pm\infty$, so, for instance, we consider $(-\infty, \infty)$ to be an open interval. Suppose c is a number, and suppose $a < c < b$. A *punctured open interval around* c is a set of all $x \neq c$ such that $a < x < b$. In other words, it is an open interval around c without c itself. Notice that a punctured open interval is the union of two open intervals.

First, let us recall the definition of a limit. (This is equivalent to Stewart 2.4.2).

Definition 1. *Let $a \in \mathbb{R}$ be a constant, and let f be a function which is definined on a punctured open interval around a.*[1] *We say that $f(x)$ has the limit $L \in \mathbb{R}$ as x approaches a if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever x satisfies $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. We write this as*

$$\lim_{x \to a} f(x) = L.$$

Importantly, the lim symbol is shorthand the entire "for every $\varepsilon > 0$ there is some $\delta > 0$ such that whenever x ..." machinery. Never forget this.

Notice that the set of all x such that $0 < |x - a| < \delta$ is a punctured open interval around a itself, which is why we require f to be defined on a punctured open interval. Also notice that this does not require f to even be defined at a itself.

If a limit exists for a function at some point, then this means the value of the function varies arbitrarily little when we restrict x to arbitrarily small intervals around a. The choice for the variable ε comes from the word "error." Because this means "infinitesimally small" changes in x correspond to "infinitesimally small" changes in $f(x)$, somehow this captures our intuition of a function being continuous:

[1] The function may well be defined elsewhere, too. We are just wanting the function to be *at least* defined in some punctured open interval around a.

Definition 2. *Let $a \in \mathbb{R}$ be a constant, and let f be a function defined on an open interval containing a. We say f* is continuous at a *if* $\lim_{x\to a} f(x) = f(a)$.

If there is an open interval containing a, then there is also a punctured open interval around a, so the limit makes sense, and $f(a)$ makes sense because the definition requires that f be defined at a.

Probably closer to most people's intuition is the following definition:

Definition 3. *A function f is* continuous on an open interval (a, b) *if it is continuous at each point in (a, b).*[2]

This is roughly equivalent to saying that a function is continuous if its graph can be drawn without lifting the pen.

With these definitions out of the way, we will prove a sequence of theorems that will be combined to ultimately show that polynomials are continuous on $(-\infty, \infty)$, as one would expect if one has ever seen the graphs of polynomials before.

Theorem 1. *Let c and a be real numbers. Then the constant function $f(x) = c$ is continuous at a.*

Proof. Assume $\varepsilon > 0$ is any real number. Let $\delta = 1$.[3] Then, whenever x is such that $0 < |x - a| < \delta$, we have $|f(x) - f(a)| = |c - c| = 0 < \varepsilon$. Since ε was arbitrary, we have found a $\delta > 0$ for every $\varepsilon > 0$, so this means $\lim_{x\to a} f(x) = f(a)$ (by the definition of the limit), and so f is continuous at a. □

Notice that this theorem works for any a, so it follows that the constant function is continuous on the entire open interval $(-\infty, \infty)$, too.

Theorem 2. *Let a be a real number. Then the function $f(x) = x$ is continuous at a.*

Proof. Assume $\varepsilon > 0$ is any real number. Let $\delta = \varepsilon$.[4] Then whenever x is such that $0 < |x - a| < \delta$, it is also the case that $|f(x) - f(a)| = |x - a| < \varepsilon$ (since $\varepsilon = \delta$). By the definition of the limit, since ε was arbitrary, this means $\lim_{x\to a} f(x) = f(a)$, and so f is continuous at a. □

Again, this theorem implies $f(x) = x$ is continuous on $(-\infty, \infty)$.

Theorem 3 (Addition rule)**.** *Let a be a real number, and let f, g be two functions defined on a punctured open interval of a such that* $\lim_{x\to a} f(x) = L_f$ *and* $\lim_{x\to a} g(x) = L_g$ *(i.e., both of these limits exist, and let L_f and L_g be the values of the limits). Then* $\lim_{x\to a}(f(x) + g(x)) = L_f + L_g$.

Proof. Assume $\varepsilon > 0$ is any real number. Since f and g approach L_f and L_g, respectively, as x approaches a, we may obtain from the definition of the limit $\delta_f > 0$ and $\delta_g > 0$ such that

[2] This makes sense because (a, b) is itself an open interval containing each point of (a, b).

[3] This is an arbitrary choice. We could have let it be ∞, if that were a number.

[4] Actually, we can let δ be any number greater than 0 but less than ε. We just need to find some punctured open interval around a on which the value of $f(x)$ doesn't get more than ε away from $f(a)$, and $\delta = \varepsilon$ is simple and sufficient.

- whenever x is such that $0 < |x-a| < \delta_f$ then $|f(x) - L_f| < \frac{\varepsilon}{2}$, and
- whenever x is such that $0 < |x-a| < \delta_g$ then $|g(x) - L_g| < \frac{\varepsilon}{2}$.

The "epsilon" from which we get δ_f and δ_g is $\frac{\varepsilon}{2}$ in this case. Let $\delta = \min\{\delta_f, \delta_g\}$. So, whenever x is such that $0 < |x-a| < \delta$, by the triangle inequality[5] and substitution,

$$\begin{aligned}|(f(x)+g(x)) - (L_f + L_g)| &= |(f(x) - L_f) + (g(x) - L_g)| \\ &\leq |f(x) - L_f| + |g(x) - L_g| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, since ε was arbitrary, there is a δ for every $\varepsilon > 0$, so the limit of $f(x) + g(x)$ as x appoaches a exists, and it is $L_f + L_g$. □

Do not confuse this theorem with its converse. It is not generally true in that if $\lim_{x\to a}(f(x) + g(x))$ exists, then so does both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$. For instance, consider $\lim_{x\to a}(\frac{1}{x} + \frac{-1}{x})$.

Notice that the theorem implies that if f and g are each continuous at a, then the limit of $f(x) + g(x)$ as x approaches a is $f(a) + g(a)$.[6]

The next theorem is fairly tricky. Try to notice all of the tricks: the triangle inequality, adding clever forms of 0, and choosing "epsilons" so that there is a δ which makes everything work out perfectly in the end.

Theorem 4 (Product rule)**.** *Let a be a real number, and let f, g be two functions defined on a punctured open interval of a such that $\lim_{x\to a} f(x) = L_f$ and $\lim_{x\to a} g(x) = L_g$. Then $\lim_{x\to a} f(x)g(x) = L_f L_g$.*

Proof. Assume $\varepsilon > 0$ is any real number. By existence of the limits for f and g as x approaches a, we can obtain δ_f, δ_g, and δ_0 with the following properties:

- Whenever x is such that $0 < |x-a| < \delta_f$, then $|f(x) - L_f| < \frac{\varepsilon}{2(1+|L_g|)}$.
- Whenever x is such that $0 < |x-a| < \delta_g$, then $|g(x) - L_g| < \frac{\varepsilon}{2(1+|L_f|)}$.
- Whenever x is such that $0 < |x-a| < \delta_0$, then $|g(x) - L_g| < \varepsilon$.

Let $\delta = \min\{\delta_f, \delta_g, \delta_0\}$. Then, whenever x is such that $0 < |x-a| < \delta$, we have that

$$\begin{aligned}|f(x)g(x) - L_f L_g| &= |(f(x)g(x) - L_f g(x)) + (L_f g(x) - L_f L_g)| \\ &\leq |f(x)g(x) - L_f g(x)| + |L_f g(x) - L_f L_g| \\ &= |g(x)||f(x) - L_f| + |L_f||g(x) - L_g|.\end{aligned}$$

[5] This is the statement that $|a+b| \leq |a| + |b|$. This is true if a, b are either real or complex. If you need convincing it is true for the real case, go through all the possibilities for a, b being positive or negative.

[6] I am overlooking one thing that I will leave as an exercise. Why is the limit unique? In other words, show that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$. No, we do not already know this, and yes, if we want to be careful, we ought to prove uniqueness.

Notice that

$$\begin{aligned}|g(x)| &= |(g(x) - L_g) + L_g| \\ &\le |g(x) - L_g| + |L_g| \\ &< 1 + |L_g|,\end{aligned}$$

because $\delta \le \delta_0$. Also notice that $|L_f| < 1 + |L_f|$. So, replacing terms from above with things that are at least as large, we have

$$\begin{aligned}|g(x)||f(x) - L_f| + |L_f||g(x) - L_g| &< (1 + |L_g|) \cdot \frac{\varepsilon}{2(1 + |L_g|)} + (1 + |L_f|) \cdot \frac{\varepsilon}{2(1 + |L_f|)} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence, $|f(x)g(x) - L_f L_g| < \varepsilon$. Because ε was arbitrary, we have a $\delta > 0$ for every $\varepsilon > 0$, so the limit exists and is $L_f L_g$. □

Now, let us put these general theorems together for some more specific statements. These are corollaries because they follow almost immediately from the theorems.

Corollary 1. *For $n \ge 0$ a natural number and a a real number, $\lim_{x \to a} x^n = a^n$. That is, $f(x) = x^n$ is continuous at a.*

Proof. Intuitively speaking, write x^n as $x \cdot \cdots \cdot x$ and repeatedly apply Theorem 4, using Theorem 2 to say $\lim_{x \to a} x = a$.

More formally, we will prove this by induction.[7] For $n = 0$, we use Theorem 1 with $c = 1$ to say $\lim_{x \to a} x^0 = \lim_{x \to a} 1 = 1 = a^0$. Now, assume $n > 0$ and that we have already proved $\lim_{x \to a} x^{n-1} = a^{n-1}$. We can write $x^n = x \cdot x^{n-1}$, so by Theorem 4, $\lim_{x \to a} x \cdot x^{n-1} = (\lim_{x \to a} x)(\lim_{x \to a} x^{n-1}) = a \cdot a^{n-1} = a^n$, since both of these limits exist, which we know by Theorem 2 and by the inductive hypothesis. Hence, $\lim_{x \to a} x^n = a^n$. This completes the induction. □

Without the product rule at our disposal, imagine trying to prove x^n is continuous at any a! General theorems make our lives so much easier!

Corollary 2. *Let a and c be real numbers, and f be a function with $\lim_{x \to a} f(x) = L_f$. Then $\lim_{x \to a} cf(x) = cL_f$.*

Proof. By Theorem 1, $\lim_{x \to a} c = c$, so by the product rule, $\lim_{x \to a} cf(x) = cL_f$. □

We are basically done. As an exercise, using induction, one may show that $\lim_{x \to a}(f_1(x) + f_2(x) + \cdots + f_n(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + \cdots + \lim_{x \to a} f_n(x)$. This makes the following corollary easy:

[7] The idea with induction is that, if you prove something for $n = 0$, and you prove it for some n given the assumption it is true for $n - 1$, then you have proved it for all n. Imagine chaining all of the arguments together in a list for $n = 0, 1, 2, \ldots$.

Corollary 3. *Suppose* $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ *is some arbitrary polynomial, whose coefficients* c_i *are real numbers. Let* a *be a real number. Then* $\lim_{x \to a} p(x) = p(a)$. *That is,* p *is continuous at* a.

Proof. This follows from the extended addition rule and the corollaries above. In fact, this proves that all polynomials are continuous on the interval $(-\infty, \infty)$. □

Hopefully this note at least gives some perspective on the structure of ε-δ-style proofs, as well as the power of proving general theorems so that specific examples become easy.

Remember, to prove that a limit exists, assume $\varepsilon > 0$ is given, write down some $\delta > 0$, then prove that whenever $0 < |x - a| < \delta$ that $|f(x) - L| < \varepsilon$. You may have to work backwards to derive your δ, but to be a logically valid proof it *must* follow the order as specified in the definition of the limit.

Exercises

Do not be discouraged by these exercises if they seem hard, because most are harder than introductory calculus expects one to be able to do at this point.

1. Write down what it means for a function f *not* to have a limit at a. Think carefully about what happens to each "for every" and "there exists."

2. Prove that the the step function, which is defined to be 0 for $x < 0$ and 1 for $x \geq 0$), does not have a limit as x approaches 0. Do this using Definition 1, not by reasoning about left- and right-sided limits.

3. Follow through the proofs of the theorems to extract a $\delta > 0$ so that $|cx - ca| < \varepsilon$ whenever x is such that $0 < |x - a| < \delta$.

4. Do the same as in the previous problem, but with $f(x) = cx^2$ instead. What about $f(x) = x^n$ (at least for various small n)?

5. Show the negation rule: if $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} -f(x) = -L$.

6. Prove that limits are unique if they exist.

7. Find a counterexample for the converse of the product rule: find f, g such that $\lim_{x \to a} f(x)g(x)$ exists but either $\lim_{x \to a} f(x)$ or $\lim_{x \to a} g(x)$ does not.

8. Prove by induction that the limit of the sum of an arbitrary number of functions is the sum of the limits of those functions, provided the limits of the functions exist.

9. Show that rational functions (i.e., quotients $\frac{f}{g}$ of polynomials) are continuous at all points where $g(x) \neq 0$. Hint: first prove the reciprocal rule, that when $\lim_{x \to a} g(x) = L \neq 0$, then $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{L}$.

10. (Extreme Value Theorem). Show that a continuous function f defined on some open interval containing $[0, 1]$ (all x such that $0 \leq x \leq 1$) has a maximum value. That is, there is an $x \in [0, 1]$ such that for all $z \in [0, 1]$, $f(z) \leq f(x)$.