THE UNION OF FLAT \((n-1)\)-BALLS IS FLAT IN \(\mathbb{R}^n\)

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**Theorem.** Let \(\beta_1^{n-1}\) and \(\beta_2^{n-1}\) be two locally flat \((n-1)\)-balls in \(\mathbb{R}^n\) with \(\beta_1 \cap \beta_2 = \partial \beta_1 \cap \partial \beta_2 = \beta^{n-2}\), where \(\beta^{n-2}\) is an \((n-2)\)-ball which is locally flat in \(\partial \beta_1\) and \(\partial \beta_2\). Then \(\beta_1 \cup \beta_2\) is a flat \((n-1)\)-ball in \(\mathbb{R}^n\).

This result has been announced by Černavskij [1], but only for \(n \geq 5\) since his outlined proof uses engulfing. Our proof avoids engulfing and works for all \(n\); a thorough knowledge of Cantrell and Lacher's version (see [2, §§4 and 5]) of Černavskij's theorem is necessary to understand our proof.

We also have another proof of the following corollary which appears in [4].

**Corollary.** Let \(g: M^{n-1} \rightarrow N^n\) be an imbedding of an \((n-1)\)-manifold into an \(n\)-manifold which is locally flat except on a set \(E\). If \(n > 3\), then \(E\) contains no isolated points (see [3] for the same result when \(M\) and \(N\) are spheres).

**Proof.** Let \(C\) be a neighborhood of an isolated point \(p\) in \(M\) which is homeomorphic to an \((n-1)\)-ball, with \(g\) locally flat on \(C - p\). Then split \(C\) into \((n-1)\)-balls \(C_1\) and \(C_2\) so that \(C = C_1 \cup C_2\) and \(C_1 \cap C_2\) is an \((n-2)\)-ball containing \(p\). \(g\) is locally flat on \(C_1\) and \(C_2\) except at the point \(p\) on their boundaries. Then, since \(n > 3\), \(g\) is flat on all of \(C_1\) and \(C_2\) by [5]. It follows from the theorem that \(C_1 \cup C_2 = C\) is flat, so \(E\) has no isolated points.

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2 Added in proof. Černavskij has independently proven this theorem by similar methods.
Let $R^n$ be Euclidean $n$-space, $B^n$ be the unit $n$-ball, and $R^k$ be imbedded in $R^n$ as $R^n = \{ x \in R^n | x_{n+1} = \cdots = x_n = 0 \}$. We will coordinatize $R^n$ by using $R^n = R^{n-1} \times R^1$ with polar coordinates on $R^1$. Thus points of $R^n$ will be triples $(s, r, \theta)$ with $s \in R^{n-1}$, $r \geq 0$, and $\theta \in R$ and with the convention that $(0, r, 0)$ is a point on the positive $x_{n-1}$-axis and $(0, r, \pi/2)$ is a point on the positive $x_n$-axis. Let $H_{\theta} = \{ (s, r, \theta) \in R^n | \theta = \phi \}$ and $D_{\theta} = H_{\theta} \cap B^n$. Note that $D_{\theta} \cup \bar{D}_{\theta} = B^{n-1}$ and $D_{\theta} \cap \bar{D}_{\theta} = B^{n-2}$. Let $W(\theta_1, \theta_2)$ be the wedge $\{ (s, r, \theta) | \theta_1 \leq \theta \leq \theta_2 \}$ and $\overline{W(\theta_1, \theta_2)} = \bar{W}(\theta_1, \theta_2) \cap B^n$.

**Proof of Theorem.** Suppose $\beta_1$ and $\beta_2$ are given by imbeddings $f_1: D_{\theta} \rightarrow R^n$ and $f_2: D_{\theta} \rightarrow R^n$. Since $\beta^{n-2}$ is locally flat in $\partial B^n\setminus D_{\theta}$, the closures of $\partial f_1(\beta^{n-2})$ and $\partial f_2(\beta^{n-2})$ are homeomorphic to $(n-1)$-balls. Then we may assume that $f_1(D_{\theta}) \cap f_2(D_{\theta}) = f_1(B^{n-1}) = f_2(B^{n-1}) = \beta^{n-2}$.

Since locally flat imbeddings of balls are flat, $f_1$ and $f_2$ extend to imbeddings of $R^n$ into $R^n$ (still called $f_1$ and $f_2$). We can require that the extensions are chosen so that $f_1(H) \cap f_2(D_{\theta}) = \beta^{n-2}$ and $f_2(B^n) \subset f_1(R^n)$. Then it suffices to show that $D_{\theta} \cap \overline{f_1 f_2}(D_{\theta})$ is locally flat. Let $f_3 = f_1 f_2$.

Since $f(D_{\theta}) \cap H_{\theta} = B^{n-1}$, we can assume that $f(D_{\theta}) \subset W(0, \pi/4)$ by rotating $f(D_{\theta})$ around $R^{n-3}$ and away from $H_{\pi/4}$, while fixing $H_{\pi/4}$. Then, in the coordinates of $f(B^n)$, we can rotate $f(D_{\theta})$ close to $f(D_{\theta})$, so we may as well assume that $f(D_{\theta}) \subset W(0, \pi/4)$ and lies between $H_{\pi/4}$ and $f(D_{\theta})$ (see Figure 1).

Let $h: R^n \rightarrow R^n$ — int $W(0, \pi/2)$ be the obvious homeomorphism which takes the wedge $W(0, \pi) - \text{int} H_{\pi/4}$ onto $W(\pi/2, \pi) - \text{int} H_{\pi/4}$ and fixes int $W(\pi, 2\pi)$. The set $W(0, \pi) \cap f(B^n)$ is separated
into two sets by \(f(D_{\pi/2})\); let \(T\) denote the set containing \(f(D_0)\). Then (see Figure 2) define an imbedding \(h: f(B^n - D_{\pi/2} \cup B^{n-2}) \to \mathbb{R}^n\) by

\[
h(f(x)) = \begin{cases} f(x) & \text{if } f(x) \in T, \\ hf(x) & \text{if } f(x) \notin T. \end{cases}
\]

To ensure that \(h\) is an imbedding it may be necessary to trim away part of \(f(B^n)\), still leaving a "ball-neighborhood" of \(f(D_0)\) (in Figure 3, restricting to the dotted ball would eliminate the annoying feelers).

Note that \(hf = f\) on \(D_0\) and \(hf(D_{\pi/2}) \subset W(\pi/2, \pi)\).

We need to extend \(hf| \bar{W}(\pi, 2\pi)\) to an imbedding of \(B^n\) into \(\mathbb{R}^n\). We can assume that for some \(\varepsilon > 0\), \(f(D_{\pi-\varepsilon}) \subset W(0, \pi/2, 2\pi)\), so then \(hf = f\) on \(D_{\pi-\varepsilon}\). Let \(g_1\) be the homeomorphism of \(B^n - D_{\pi/2} \cup B^{n-2}\) which fixes points outside \(\bar{W}(3\pi/4, 2\pi)\) and moves \(D_{\pi-\varepsilon}\) to \(D_{\pi/2}\). Let \(g_2: hf(\bar{W}(3\pi/4, 2\pi - \varepsilon)) \to hf(\bar{W}(3\pi/4, \pi))\) be the homeomorphism defined by \(g_2 = hf g_1(hf)^{-1}\). Now define an imbedding \(g: f(\bar{W}(0, 2\pi - \varepsilon)) \to \mathbb{R}^n\) by

\[
g(x) = \begin{cases} g_2(x) & \text{if } x \in hf(\bar{W}(3\pi/4, 2\pi - \varepsilon)), \\ x & \text{otherwise}. \end{cases}
\]

To make sure that \(g\) is well defined, it may be necessary to again shrink \(f(B^n)\) towards \(B^{n-2}\) so that int \(f(\bar{W}(0, 2\pi - \varepsilon)) \cap \partial hf(\bar{W}(3\pi/4, 2\pi - \varepsilon)) \subset hf(D_{\pi/2})\). Let \(i: \bar{W}(0, \pi) \to \bar{W}(0, 2\pi - \varepsilon)\) and note that \(gfi = hf\) on \(D_{\pi/2}\).

Then (see Figure 4), we can piece together \(gfi\) and \(hf\) to get an imbedding \(F: B^n \to \mathbb{R}^n\); specifically, let

\[
F(x) = \begin{cases} gfi(x) & \text{if } x \in \bar{W}(0, \pi), \\ hf(x) & \text{if } x \in \bar{W}(\pi, 2\pi). \end{cases}
\]

\(F = f\) on \(D_0\), so \(F(D_0) \subset W(-\pi/2, \pi/2)\), and \(F(D_{\pi/2}) = hf(D_{\pi/2})\).
\[ \mathcal{W}(\pi/2, 3\pi/2) \] is "transverse" to \( H_{s/3} \cup H_{s/3} \), and that is the key to the proof. It allows us to find an isotopy making \( F(D_\theta) \) tangent to \( H_\theta \) at \( T^{n-1} \) for all \( \theta \). This isotopy is constructed in the latter part of the proof of Lemma 5.2 of [2]. Then a homeomorphism of \( R^n \) can be constructed which fixes \( D_\theta \) and takes \( F(D_\theta) \) to \( D_\theta \) (see the proof of Theorem 6.1 in [2]). Thus (\( R^n, \beta_1 \cup \beta_n \)) is pairwise homeomorphic to \( (R^n, D_\theta \cup D_n) \), finishing the proof.

References


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