SMOOTHING LOCALLY FLAT IMBEDDINGS

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The fundamental imbedding problem for manifolds is to classify the imbeddings of an n-manifold into a q-manifold under ambient isotopy. We announce here that the differentiable and topological cases of this problem for differentiable manifolds are the same if \( 2q > 3(n+1) \) and \( q \geq 8 \).

This follows from Theorem 2 below which states that a locally flat imbedding of a compact differentiable manifold \( M^n \) into a differentiable manifold \( \mathbb{Q}^q \) is ambient isotopic to a differentiable imbedding if \( 2q > 3(n+1) \) and \( q \geq 8 \). Since this ambient isotopy may be chosen arbitrarily close to the identity map, the set of differentiable imbeddings is dense in the set of locally flat imbeddings of \( M^n \) in \( \mathbb{Q}^q \).

It will then follow that two locally flat imbeddings of \( M^n \) into \( \mathbb{Q}^q \) are ambient isotopic if they are homotopic; hence the classification problem reduces to a problem in homotopy theory.

**Theorem 1.** Let \( f : B^n \to \text{int} \mathbb{Q}^q \) be a locally flat imbedding of the unit n-ball into \( \mathbb{Q}^q \). Such an \( f \) always extends to \( f : \text{int} \mathbb{R}^n \to \text{int} \mathbb{Q}^q \). Let \( C^{n-1} \) be a compact differentiable submanifold of \( \partial B^n = S^{n-1} \), and suppose that \( f \) is differentiable on a neighborhood of \( C^{n-1} \) in \( B^n \). Let \( q \geq 7 \), \( 2q > 3(n+1) \) and \( \epsilon > 0 \). Then there exists an ambient \( \epsilon \)-isotopy \( F_t : \mathbb{Q}^q \to \mathbb{Q}^q \), \( t \in [0, 1] \), satisfying

1. \( F_0 = \text{identity} \),
2. \( F_t f \) is differentiable on \( \text{int} B^n \) and on a neighborhood of \( C^{n-1} \) in \( B^n \),
3. \( F_t = \text{identity on } Q - N_\epsilon(f(B^n)) \) and on \( f(\text{int} B^n) \) for all \( t \in [0, 1] \),
4. \( |F_t(x) - x| < \epsilon \) for all \( x \in Q^q \) and \( t \in [0, 1] \). \( (N_\epsilon(X)) \) is the set of points within \( \epsilon \) of \( X \).

**Theorem 2.** Let \( f : M^n \to \mathbb{Q}^q \) be a locally flat imbedding such that either \( f(M^n) \subset \text{int} \mathbb{Q}^q \) and \( q \geq 7 \) or \( f^{-1}(\partial \mathbb{Q}^q) = \partial M^n \) and \( q \geq 8 \). Let \( 2q > 3(n+1) \) and \( \epsilon > 0 \). Then there exists an ambient \( \epsilon \)-isotopy \( F_t : \mathbb{Q}^q \to \mathbb{Q}^q \), \( t \in [0, 1] \), satisfying

1. \( F_0 = \text{identity} \),
2. \( F_t f \) is a differentiable imbedding.

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1 This is an announcement of a portion of the author's dissertation at the University of Chicago written under Professor Eldon Dyer.
(3) \( F_t = \text{identity on } Q - N_t(f(M^n)) \) for all \( t \in [0, 1] \).
(4) \( |F_t(x) - x| < \varepsilon \) for all \( x \in Q^s \) and \( t \in [0, 1] \).

The proof follows from Theorem 1 by considering the handlebody decomposition of \( M^n \), and smoothing the imbedding of one handle at a time.

Only imbeddings of \( M^n \) into \( Q^s \) satisfying \( f(M^n) \subset \text{int } Q^s \) or \( f^{-1}(\partial Q^s) = \partial M^n \) will be considered. Let \( T \) be the set of equivalence classes of locally flat imbeddings of \( M^n \) into \( Q^s \) under equivalence by ambient isotopy. Similarly, let \( D(C) \) be the set of equivalence classes of differentiable (combinatorial) imbeddings of \( M^n \) into \( Q^s \) under equivalence by ambient diffeotopy (ambient combinatorial isotopy). Let \( H \) be the homotopy classes of locally flat imbeddings of \( M^n \) into \( Q^s \). \( H \) is a subset of \([M^n, Q^s]\), the homotopy classes of maps of \( M^n \) into \( Q^s \). Then we have the following commutative diagram where the maps are the natural projections.

\[
\begin{array}{ccc}
D & \xrightarrow{\pi} & M^s, Q^s \\
\downarrow & & \downarrow i \\
T & \xrightarrow{\beta} & H \\
\downarrow & & \downarrow \gamma \\
C & \xrightarrow{\rho} & [M, Q] \\
\end{array}
\]

\( \beta \) is clearly onto for all \( n \) and \( q \). Gluck has shown [1] that \( \rho \) and \( \gamma \), and hence \( \beta \) and \( \beta^i \) are isomorphisms for \( q \geq 2n + 2 \). Haefliger has shown [2] that \( \pi \) is a monomorphism and that \( \alpha \) is an isomorphism if \( 2q > 3(n + 1) \).

It follows from Theorem 2 that \( \pi \) is also epimorphic if \( 2q > 3(n + 1) \) and either \( q \geq 7 \) when \( f(M^n) \subset \text{int } Q^s \) or \( q \geq 8 \) when \( f^{-1}(\partial Q^s) = \partial M^n \). Then \( \pi \) and \( \beta \) are isomorphisms in this range of dimensions.

REFERENCES

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