CODIMENSION-TWO LOCALLY FLAT EMBEDDINGS HAVE NORMAL BUNDLES

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Let \( P^p \) and \( Q^q \) be topological manifolds of dimensions \( p \) and \( q \), respectively, and let \( i: P \to Q \) be a locally flat imbedding. If \( p + 1 = q \) and \( i(P) \) separates \( Q \), then Brown has shown that \( i(P) \) is flat, i.e., it has a trivial normal bundle \([1]\).

We have a similar theorem in codimension 2.

**Theorem 1.** If \( p + 2 = q \), \( \exists \) a neighborhood \( B \) of \( i(P) \) and a map \( \pi: E \to i(P) \) which is a bundle \( v \) with fiber \( B^2 \) and structural group \( \mathcal{H}(R^3) \), the space (with the CO-topology) of homeomorphisms of \( R^2 \) which fix the origin, \( v \) is unique up to ambient isotopy.

If \( \partial P \neq \emptyset \) and \( P = P \cup (\partial P \times [0, 1]) \), then \( i \) extends to a locally flat embedding \( \tilde{i}: \tilde{P} \to Q \), where \( \tilde{P} \) is a bundle over \( P \) with fiber \( \partial P \times [0, 1] \).

Since \( \mathcal{H}(R^3) \) deforms to \( O(2) \), \( \tilde{i} \) is a normal bundle.

Since \( \mathcal{H}(R^2) \cong \text{TOP}_2 \cong O(2) \cong S^1 \times S^1 \), and since there is a universal bundle \( \text{TOP}_2 \to E_{\text{TOP}_2} \to B_{\text{TOP}_2} \), with contractible total space \( E_{\text{TOP}_2} \), we see that

\[
\pi_i(B_{\text{TOP}_2}) = \begin{cases} 
0, & i \neq 1, 2, \\
Z, & i = 1, \\
Z, & i = 2.
\end{cases}
\]

The topological two-plane bundles over \( P \) are classified by maps \( P \to B_{\text{TOP}_2} \). Thus the oriented bundles over \( P \) are classified by \( H^2(P; \pi_2(\text{TOP}_2)) = H^2(P; \mathbb{Z}) \).

If \( q - p \geq 3 \), it is known that there exist locally flat embeddings (in fact, PL embeddings) which have no normal disk bundles \([3]\). But if \( q - p \) is large enough with respect to \( p \), then normal bundles do exist \([10]\). Since normal bundles do not always exist, it would be nice to have normal block bundles and a good topological block bundle theory \([10, 11]\).

However, topological block bundles will have to differ somewhat from PL block bundles because there are topological manifolds without handlebody structures in dimension four or five \([7]\).

Section 1 contains definitions and background, Section 2 has the main lemma, Theorem 1 is proved in Section 3, and Section 4 has an application on straightening handles.

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Let \( R^p \) be euclidean n-space, \( R^p = \{ x \in R^p | x \geq 0 \} \), \( rB^p \) the ball of radius \( r \) in \( R^p \), \( rS^{p-1} \) its boundary, and \( B^p \) its interior.

\( i: P \to Q \) is said to be locally flat if for each \( \bar{p} \in Q \), there exists a neighborhood \( N \) such that \( (N, \partial N \cap i(P)) \) is pairwise homeomorphic to \((R^p, \partial R^p)\), where \( p \) and \( q \) are the dimensions of \( P \) and \( Q \). \( i(P) \) is flat if \( i \) extends to an embedding \( i: P \times R^p \to Q \); i.e., \( i(P) \) has a trivial normal bundle. If \( \partial P \neq \emptyset \) and \( i(P) \cap \partial Q \), \( i \) is locally flat (flat) if \( i \) extends to a locally flat (flat) embedding of \( P \cup (\partial P \cap \partial Q) \). This condition is equivalent to \((N, \partial N \cap i(P)) \) being pairwise homeomorphic to \((R^p, \partial R^p)\) for \( p \in \partial P \). If \( i \) is proper (proper \((\partial i(C) = \partial P) \), then it is locally flat if \( (N, \partial N \cap i(P)) \) is homeomorphic to \((R^p, \partial R^p)\) for \( p \in \partial P \).

Let \( \mathcal{H}(X) \) be the space (with the compact open topology) of homeomorphisms of \( X \) which fix \( Y \) pointwise. A basis for the neighborhoods of the identity consists of sets of the form \( N(C, \epsilon) = \{ h \in \mathcal{H}(X) | d(h, x) < \epsilon \} \) for all \( C \) and all compact sets \( C \) and \( \epsilon > 0 \).

The following statements can be found in \([2]\). If \( L \) is locally flat in \( M \) and both \( L \) and \( M \) are compact or interiors of manifolds with boundaries, then \( \mathcal{H}(L) \) is locally contractible. Let \( J \) and \( K \) be compact subsets with \( J \subset \text{int} K \subset M \). Given \( e > 0 \), there exists \( \delta > 0 \) such that if \( h \in N(K, \delta) \), then \( \exists \) a canonical isotopy \( h_t: M \to M, t \in [0, 1] \) with \( h_0 = h, h_1 = \text{identity}, h_t \in N(K, \delta) \), and \( h_t \to h_0 \) as \( t \to 0 \).

Let \( g: L \to M \) be a locally flat isotopy; i.e., \( G = (g, \epsilon) : L \times I \to M \times I \) is locally flat in a level-preserving way. Then \( g \) extends to an ambient isotopy of \( M \). Further, if \( g \) is small, then so is the extension and it is supported on a neighborhood of \( G(L \times I) \).

We will say that \( A \) is a weak deformation retract of \( X \) if \( A \) is a homotopy \( H_1 : X \to X, t \in [0, 1] \), with \( H_0 = \text{identity}, H_1 \in N(A, \epsilon) \subset A, H_1(A) \subset A \).

2

The main new idea in Theorem 1 is contained in the next lemma. Then by applying it, using standard techniques, we get Theorem 1.

**Lemma.** Let \( h: M \times R^2 \to M \times R^2 \) be a homeomorphism with \( h(M \times 0) = \text{id} \), where \( M \) is a compact manifold. Then \( h \) is isotopic to a fiber

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preserving homeomorphism. Specifically, \( h_t : M \times R^2 \to M \times R^2, t \in [0, 1] \) with \( h = h_0, h_t(0) = 0 \) for all \( t \) and \( h_t(z \times R) = z \times R \) for all \( z \in M \).

Proof. We will isotope \( h \) so that it becomes close enough to a rotation so as to apply local contractibility to move \( h \) to the rotation (= homeomorphism \( \rho : M \times R^2 \to M \times R^2 \) with \( \rho(z, \theta) = (z, \rho_\theta(z)) \)).

Using polar coordinates for \( R^2 \), we describe points of \( M \times R^2 \) by triples \( (z, \theta, t), z \in M, (\theta, t) \in R^2 \); \( h \) can be written \( h(z, \theta, t) = (h(z, \theta, t), h_\theta(z, \theta, t), h_t(z, \theta)) \). We will isotope \( h_\theta \) and \( h_t \) so that they are small enough (on \( M \times K^B \)) for some \( K > 0 \) and \( h_t \) so that it is close enough to a rotation.

Step 1. There is a well-known argument for making \( h_\theta \) small. Let \( C_\theta = M \times tS^1 \) and \( D_{\theta} = M \times tB^2 \). We can assume by squeezing that \( D_{\theta} \subset h(D_{\theta}) \subset D_{\theta} \), and \( h(D_{\theta}) \subset D_{\theta} \), (see Figure 1). We need to move \( h(C_\theta) \) out of \( C_{\theta} \) and \( C_{\theta} \), without moving \( h(C_\theta) \). For a small enough \( r > 0 \), \( h(C_\theta) \subset D_{\theta} \). We use the radial structure given by \( h \) to obtain a homeomorphism \( f : M \times R^2 \to M \times R^2 \) which slides \( h(C_\theta) \) to \( h(C_\theta) \) and fixes \( h(C_\theta) \) so that \( f(D_{\theta} \setminus C_{\theta}) \subset h(D_{\theta}) \). We use the radial structure given by \( f \) to slide \( h(C_\theta) \) close enough to \( C_{\theta} \) so that it is between \( C_{\theta} \) and \( C_{\theta} \). Now \( D_{\theta} \subset h(D_{\theta}) \subset D_{\theta} \subset h(D_{\theta}) \subset D_{\theta} \).

![Figure 1](image)

We may iterate this process countably many times so that \( D_{\theta} \subset h(D_{\theta}) \subset D_{\theta} \), where the sequence \( K = t_1 > t_2 > \ldots > 0 \) forms an arbitrarily fine subdivision of \( [0, K] \) and the \( e_i \) are as small as desired. \( h_\theta \) is now small on \( M \times K^B \). Call this new homeomorphism \( h' \).

Step 2. Let \( h'(x, 0, t) = (h'(x, 0, t), h_t(x, 0, t), (1/\delta)h_t(x, 0, \delta t)) \). We claim that if \( \delta \) is small enough, then \( h'_t \) is arbitrarily small on \( M \times K^B \), and that \( h'_t \) is still small enough. The first follows because \( h' \) is continuous and is the identity on \( M \times 0 \), and the second follows if \( \{t_1\} \) is fine enough.

Step 3. We will isotope \( h'' \) so that on \( M \times K^B \) it is close enough to the rotation \( \rho \), defined by \( \rho(z, \theta, t) = (z, \theta + h'_t(z, \theta, t)) \). Let \( g = \rho^{-1} h'' \). If \( h'_t \) and \( h'_t \) are small enough on \( M \times K^B \), then \( g \) will be small enough near \( M \times 0 \times (0, K) \). In particular, given \( e > 0, 3 \delta > 0 \) such that \( g(x, 0, t) \in M \times [-e, e] \times (0, \infty) \) for \( t \in [-\delta, \delta] \) and \( t \in (0, K] \) (see Figure 2).

![Figure 2](image)

Let \( g \) be a certain finite covering of \( g \):

\[
\begin{align*}
M \times R^2 &\xrightarrow{\lambda} M \times R^2 \\
\lambda &\xrightarrow{\delta} M \times R^2 \\
M \times R^2 &\xrightarrow{\delta} M \times R^2 \\
\end{align*}
\]

Let \( \lambda : S^1 \to S^1 \) be defined by \( \lambda_\delta(0) = n\delta(2\pi) \), the \( n \)-fold covering map. Let \( \lambda_\delta \) be an approximation to \( \lambda_\delta \), with the properties that \( \lambda_\delta = id \) on \([-e, e]\) and \( \lambda_\delta = \lambda_\delta \) outside \([-2e, 2e]\), where we need to have chosen \( e \) small enough.

Finally, let \( \lambda = id \) on \( M \times 0 \) and \( \lambda(x, 0, t) = (x, \lambda_\delta(\theta), t) \) for \( t > 0 \). Then it is
easy to see that \( g \) lifts to a homeomorphism \( \tilde{g} \) with \( g = \tilde{g} \) on the wedge \( M \times [-\delta, \delta] \times (0, K) \). Therefore, \( g \) is isotopic to \( \tilde{g} \), via an isotopy fixing \( M \times 0 \).

It is not hard to verify that we can make \( \tilde{g} \) arbitrarily small on \( M \times K^2 \) by taking \( \delta \) large enough. By local contractibility, \( \tilde{g} \) is isotopic to the identity on \( M \times K^2 \), and hence on \( M \times R^2 \).

Since \( \rho^{-1} h = \tilde{g} \) is isotopic to the identity, it follows that \( \tilde{h} \) and, therefore \( h \), is isotopic to the rotation \( \rho \) (fixing \( M \times 0 \) throughout), finishing the proof of the lemma.

**Remark 1.** Suppose \( h \) is fiber preserving on a neighborhood of a subset \( L \) of \( M \). Then we can find an isotopy \( h_t \) with the additional property of being fiber preserving on a smaller neighborhood of \( L \) in \( M \). The isotopies constructed in steps 1, 2, and 3 are all clearly fiber preserving if \( h \) is, except possibly the isotopy constructed using local contractibility. \( \tilde{g} \) is small on each fiber over the neighborhood of \( L \), so we isotop \( \tilde{g} \) to the identity on each fiber separately. This is done in a continuous way (using local contractibility) so we get a fiber-preserving isotopy \( \tilde{g}_t : M \times R^2 \to M \times R^2 \), with \( \tilde{g}_0 = \tilde{g} \) and \( \tilde{g}_1 = \text{identity on a neighborhood of } L \) (see Section 1 on local contractibility). \( \tilde{g}_1 \) can still be small enough to be isotopic to the identity elsewhere, using the relative form of local contractibility.

**Remark 2.** If \( M \) is not compact, then we may find \( h_t \) with \( h_0 \) fiber preserving on an arbitrarily large compact subspace. All the steps in the lemma rely on compactness. For example, in step 3, \( \rho \) may wind \( M \times R^2 \) around \( M \times 0 \) more and more as one approaches the open ends of \( M \). But on a compact subset, this winding is bounded so some finite cover gives \( \tilde{g} \) is small enough.

**Remark 3.** We can require that \( h_0 \) have compact support if we only require \( h_t \) to be fiber preserving near \( M \times 0 \). Specifically, we get \( h_t = \text{id} \) outside some compact set and \( h_t(x \times K^2) = x \times K^2 \) for \( x \) in a compact subset of \( M \). One just checks that all constructions can be done in a neighborhood of \( C \times 0, C \text{ compact in } M \).

**Remark 4.** The lemma still holds when \( h \) is only an embedding (with \( h(M) = \phi(id) \)). This follows because we have noted that all constructions are done in a neighborhood of \( M \times 0 \) (or \( \partial M \not\subseteq \Omega \), we need to assume that \( h \) is a proper embedding.)

### Proof of Theorem 1

Let \( P_0 \) be an open submanifold of \( P \) with a normal bundle \( v_0 \) over \( \partial P_0 \) in \( Q \). Let \( R^2 \) be a coordinate patch in \( P \) with \( \tilde{R}^2 \) flat in \( Q \) and let \( M = P_0 \cap R^2 \). Suppose at first that \( v_0(M) \) is trivial so that there is an embedding \( \alpha : M \times R^2 \to Q \) with \( \alpha(M \times R^2) = E(v_0(M)) \). Since \( \tilde{R}^2 \) is flat, let \( \beta : M \times R^2 \to Q \) be the flat structure on \( \alpha(M) \). We can assume that \( \beta(M \times R^2) = (M \times R^2) \), so as to consider \( \alpha^{-1} \beta : M \times R^2 \to M \times R^2 \). By the lemma (including the remarks, particularly 4), \( \alpha^{-1} \beta = h _ {\text{isotopic (with compact support)}} \) to an embedding \( h_0 \) with \( h_0(x \times K^2) = x \times K^2 \) for all \( x \) in some large compact subset of \( M \). Then

\[
\beta_t = \begin{cases}
\alpha_t & \text{on } M \times R^2, \\
\beta & \text{on } (R^2 - M) \times R^2
\end{cases}
\]

is an isotopy fixing \( R^2 \) with \( \beta_t(x \times K^2) = \alpha_t(x \times K^2) \) for \( x \) in the above compact subset of \( M \). By restricting \( \beta_t \) to \( R^2 \) \( \text{int } K^2 \) and then applying the microbundles-are-bundles argument in [8], we get that \( \beta_t(x \times R^2) = \alpha_t(x \times R^2) \) for the above \( x \). Thus, taking appropriate refinements of \( P_0 \) and \( R^2 \), say \( P_0 \) and \( R_0 \), we have extended \( v_0 \) to a bundle over \( \tilde{P}_0 \cup R_0 \).

Now suppose \( v_0(M) \) is trivial. Then we cover a large enough compact subset of \( M \) with open sets \( M_0, \ldots, M_n \) on which \( v_0 \) is trivial. We proceed as above with \( M_0 \). Then for \( M_2, \ldots, M_n \) we use the relative form of the lemma in Remark 1, to make \( \alpha_t^{-1} \beta_t \) fiber preserving over large compact subsets of \( \bigcup_{j=1}^{n-1} M_j = \{ 2, \ldots, n \} \). So as before \( v_0 \) extends over \( \tilde{P}_0 \cup R_0 \).

If \( P \) is compact with \( \partial P \not\subseteq \Omega \), we construct \( v \) coordinate patch by coordinate patch, as above, using appropriate refinements; when \( P \) is open, paracompactness is sufficient for the same construction to work. This sort of argument is well known and we omit further details.

If \( \partial P \not\subseteq \Omega \) and \( \tilde{P} = \partial P \), we add an open collar to \( \partial P \), extend \( i, j \), and proceed as above. If \( i \) is proper \( \tilde{P} \cap (Q \cup \partial P) \), then we construct \( v \) on \( \partial P \) in \( \tilde{P} \cup \partial P \), extend to collars, and continue as above.

### Proposition

\( \mathcal{F}_p(R^2) \) is a weak deformation retract of \( \mathcal{F}_p - \partial P(R^2) \).

**Proof.** \( \mathcal{F}_p - \partial P(R^2) \) is clearly a strong deformation retract of \( \mathcal{F}_p(R^2) \). But by adding a point at infinity, compactifying each homeomorphism, and removing the origin, we see that there is a homeomorphism \( \Omega : \mathcal{F}_p - \partial P(R^2) \to \mathcal{F}_p(R^2) \). Now \( \mathcal{F}_p(R^2) \) deforms to \( \mathcal{F}_p(R^2) \) by Theorem 1 of [8]. Applying \( \Omega^{-1} \) to this deformation shows that \( \mathcal{F}_p(R^2) \) is a weak deformation retract of \( \mathcal{F}_p - \partial P(R^2) \).

Furthermore, if \( h_0 \in \mathcal{F}_p - \partial P(R^2) \) and \( h_1 \) is the deformation taking \( h_0 \) into \( \mathcal{F}_p(R^2) \), then \( h_1 \) has compact support. This follows directly from Kister’s proof.

### Theorem 2

Let \( h : B^2 \times R^2 \to B^2 \times R^2 \) be a homeomorphism with \( k \neq 1 \) and \( h \) is identity on \( S^{2-1} \times R^2 \). Then \( h \) is isotopic to the identity, fixing \( h_0 \).
Proof. We can assume that \( h = \text{identity} \) on \( (B^n - \frac{1}{4}B^n) \times R^2 \). Then the interior of \( B^4 \times R^2 \) is \( R^{n+2} \), and from the proposition above, it follows that \( h \) is isotopic (rel \( \partial \)) to a homeomorphism \( \tilde{h} \) which fixes \( B^3 \times 0 \). It is now easy to see that the method of proof of the lemma works here to give the desired isotopy of \( h \) to the identity (rel \( \partial \)). (Since \( \partial \tilde{h} = \text{identity} \), it is not necessary to alter it on the \( \partial \) in steps 1 or 2, since \( k \neq 1 \) and \( \pi_1(\mathbb{R}^2) = 0 \), no alteration is necessary in step 3 either.)

Now consider the problem of straightening 3-handles (see [4], [5], and [6]). Let \( h : B^3 \times R^2 \to V \) be a homeomorphism, \( \text{PL} \) on the boundary, onto a \( \text{PL} \) manifold \( V \). We wish to straighten \( h \), i.e., find an isotopy \( h_t \), \( t \in [0, 1] \), with \( h_0 = h \), \( h_t \in \text{PL} \). If \( n \geq 3 \), Sullivan has shown that \( h_t \) extends to a \( \text{PL} \) homeomorphism \( h : B^3 \times R^2 \to V \). However, 3 nonstraightenable 3-handles for \( n \geq 2 \), so 3 homeomorphisms \( g = h^{-1} : B^3 \times R^2 \to B^3 \times R^2 \), identity on \( \partial \), which are not isotopic to \( \text{PL} \) homeomorphisms rel \( \partial \), when \( n \geq 3 \).

On the other hand, when \( n = 2 \), we have just seen in Theorem 2 that any homeomorphism \( g : B^3 \times R^2 \to B^3 \times R^2 \), \( g = \text{identity} \) on \( \partial \), is isotopic to the identity, rel \( \partial \). Thus, since 3 nonstraightenable 3-handles \( h : B^3 \times R^2 \to V \), we see that \( V \) cannot be \( \text{PL} \) homeomorphic to \( B^3 \times R^2 \), rel \( \partial \). Therefore, \( B^3 \times R^2 \) has more than one \( \text{PL} \) structure rel \( \partial \) (in fact, two).

So nonstraightenable 3-handles \( h : B^3 \times R^2 \to V, n \geq 2 \), arise in two ways: if \( n = 2 \), \( V \) is not \( \text{PL} \) homeomorphic to \( B^3 \times R^2 \) rel \( \partial \) so of course \( h \) cannot be straightened; if \( n \geq 3 \), \( V \) is \( \text{PL} \) homeomorphic to \( B^3 \times R^2 \) rel \( \partial \), but the homeomorphism \( h \) is bad.

We say that two \( \text{PL} \) structures on a manifold are equivalent up to isotopy (homotopy) if the identity is isotopic (homotopic) to a \( \text{PL} \) homeomorphism.

From [4], [5], [6], and the above, the \( \text{PL} \) structures (rel \( \partial \)) up to isotopy on \( B^3 \times R^2, n \geq 2 \), correspond to \( H^1(B^3 \times R^2, \partial; Z_2) = Z_2 \), and the \( \text{PL} \) structures (rel \( \partial \)) up to homotopy correspond to \( Z_2 \) if \( n = 2, 0 \) if \( n \geq 3 \).

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References