AN EXOTIC INVOLUTION OF $S^4$

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CAPPELL and Shaneson[1] construct a family of smooth 4-manifolds which are simple homotopy equivalent to real projective 4-space $\mathbb{RP}^4$, but not even smoothly h-cobordant to $\mathbb{RP}^4$. (It is possible they are homeomorphic to $\mathbb{RP}^4$.) It is natural to ask whether their double covers are $S^4$ or not.

**Theorem.** Let $Q^4$ be the fake $\mathbb{RP}^4$ built with the matrix $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, as described below. Then the double cover of $Q^4$, called $\Sigma^4$, is diffeomorphic to $S^4$. Hence there is an exotic free involution on $S^4$.

The proof that $\Sigma^4$ is $S^4$ begins with a precise picture of the zero, one and two-handles of $\Sigma^4$. After sliding some handles over other handles, we cancel appropriate pairs of one and two-handles. The result is seen to be $S^2 \times B^2 \# S^2 \times B^2 \# S^2 \times B^2$. Their complement in $\Sigma^4$ consists of the 4-handle and three-handles, which is diffeomorphic to $S^1 \times B^3 \# S^1 \times B^3 \# S^1 \times B^3$. These pieces are glued together by a diffeomorphism $h$ of $S^1 \times S^2 \# S^1 \times S^2 \# S^1 \times S^2$ to get $\Sigma^4$. Laudenbach and Poenaru have shown[2] that $\Sigma^4$ must be diffeomorphic to $S^4$.

**Proof.** Recall the construction of $Q^4$: divide $\mathbb{RP}^4$ into two pieces; one is the normal $B^2$-bundle over $\mathbb{RP}^2$, called $\mathbb{RP}^2 \times B^2$; the other is its complement which is the non-trivial bundle over $S^4$; namely $S^1 \times B^3$. Cappell and Shaneson replace $S^1 \times B^3$ with a manifold $C$ with $\partial C = S^1 \times S^2$. Let $A: R^3 \to R^3$ be the orientation reversing linear map given by the matrix $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. (This is one of many matrices that work in $\Sigma^4$.)

Cappell and Shaneson's general construction, and the only one we consider here.) Taking quotients, we get an orientation reversing diffeomorphism $A: T^3 \to T^3$, where $T^3 = R^3/(2\mathbb{Z})$. Via a small isotopy in a neighborhood of $0 \in T^3$, we can assume $A$ is a reflection on $U = \{ x \in R^3 | |x| < \varepsilon \}$. Let $C$ be the mapping torus of $A|T^3 - V$, i.e. $C = ((T^3 - V) \times [-1,1])/(x,-1) \sim (A(x),1)$, where $V$ is a small open ball inside $U$.

It follows that the double cover $\Sigma^4$ of $Q^4$, can be constructed from the double cover of $\mathbb{RP}^2 \times B^2$ which is $S^3 \times B^2$ and the double cover of $C$ which is $\hat{C} = \{ \text{mapping torus of } A|T^3 - V \}$.

To build $\Sigma^4$ as a handlebody we construct $\hat{C}$ and then turn $S^2 \times B^2$ "upside down" and add its 2-handle and 4-handle. To get $\hat{C}$, start with $T^3 \times [-1,1]$ which has a 0-handle, three 1-handles, $\alpha_1$, $\alpha_2$, and $\alpha_3$, three 2-handles $\alpha_1$, $\alpha_2$ and $\alpha_3$ and a 3-handle. These handles are attached to the boundary of the 0-handle $B^3 \times [-1,1]$ which is pictured as $R^3 \cup \infty$ where $\partial B^3 \times 0$ is $\partial [-1,1]^3$, $0 \times 1$ is the origin of $R^3$ and $0 \times (-1)$ is $\infty$ (see Fig. 1). The 1-handle $\alpha_3 (= B^1 \times B^3)$ is attached by its ends $S^0 \times B^3$ to small 3-balls at the ends of the unit vectors on the x-axis. Similarly with $\alpha_2$ and the y-axis, and $\alpha_3$ and the z-axis. The 1-handles are not drawn, just the attaching maps. The 2-handles $\alpha_2$ are attached along the indicated circles, where the gaps are filled in by going over 1-handles. In particular, the attaching circle of $\alpha_1$ "lies in" the yz-plane, $\alpha_2$ in $zx$-plane, and $\alpha_3$ in the xy-plane. We don't need the 3-handle, and its attaching map is not drawn.
To see how the mapping torus is formed, reflect on the analogous picture for \( M^3 = (T^2 \times (2\text{-}handle)) \times [-1, 1] / (x, -1) - (B(x), 1) \) where \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). We need to isotope \( B \) so that it takes 1-handles into 1-handles. The isotopy is analogous to \( \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \), \( t \in [0, 1] \). It is drawn in Fig. 2.

In Fig. 3, we form \( M^3 \) by adding a 1-handle \( a_* \) to a small 2-ball centered at \( 0 \times (-1) \) and another at \( 0 \times 1 \). Then we add two 2-handles, \( \beta_1 \) and \( \beta_2 \) along the arcs indicated. The result is \( M^3 \).

Working analogously in one higher dimension, we isotop \( A^2 \) so that it takes 1-handles into 1-handles (Fig. 4). Since \( A^2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \), we begin with the isotopy

\[
\begin{pmatrix}
  t & 0 & 1-t \\
  -1 & 1 & 0 \\
  0 & t & 1
\end{pmatrix}, \quad t \in [0, 1],
\]

near the origin, to get the second picture in Fig. 4. In the remaining pictures, we push the 1-handles into the 1-handles.

Now we attach a 1-handle \( a_* \) to a small 3-ball centered at \( \infty \) and one centered at the origin (this connects \( 0 \times (-1) \) and \( 0 \times 1 \) in \( T^3 \times [-1, 1] \)). Then we attach 2-handles \( \beta_1, \beta_2 \) and \( \beta_3 \) along the indicated arcs in Fig. 5 (this has the effect of identifying the 1-handles in \( T^3 \times (-1) \) with their image under \( A^2 \) in \( T^3 \times 1 \)). This finishes the construction of the handles of \( \hat{C} \) up to index 2. (But note that we have slightly changed the definition of \( \hat{C} = (T^3 - V) \times [-1, 1] / A^2 \). \( V \) is no longer a small open ball centered at \( 0 \in T^3 \), but is now a "blister" on the 0-handle \( B^1 \), missing the attaching maps of the 1- and 2-handles. It is described more precisely below where the attaching map of \( \gamma \) is defined.) It would require three 3-handles to identify the three 2-handles in \( T^3 \times (-1) \) with their images in \( T^3 \times 1 \), but these are not drawn; instead they (together with the 4-handle) produce the \( S^1 \times B^3 \neq S^1 \times B^3 \# S^1 \times B^1 \) mentioned at the beginning.

Finally, we must add the dual 2-handle \( \gamma \) of the \( S^2 \times B^2 \), i.e. a thickened point \( \times B^2 \). The \( S^2 \times B^2 \) is added to \( \partial V \times [-1, 1] / A^2 = S^2 \times S^1 \), so \( \alpha \) is attached along a thickened point \( \times S^1 \), which we can assume lies on \( \partial (\text{0-handle} \cup a_* \} \). In the lower dimensional

Fig. 2.
case, Fig. 3, this 2-handle $\gamma$ would be attached along the indicated arc, which continues around the 1-handle $a_x$. In our case the arc is analogous to a ray from the origin to $\infty$, which avoids the attaching maps of other handles. For convenience, we pick a line given by the vector $(\frac{1}{2}, 1, 0)$, and connect its ends via the 1-handle $a_x$. The $S^2 \times B^2$ can be added to $\mathcal{C}$ with or without a twist on $S^1 \times S^2 = \partial \mathcal{C} = \partial (S^2 \times B^2)$. In our case there is no twist, so the framing for the attaching map for $\gamma$ is the untwisted one.
The remainder of the proof involves cancelling $a_*$ with $\gamma$ and then $a_1$, $a_2$ and $a_3$ with $\beta_1$, $\beta_2 - \beta_1$, and $\beta_3$ respectively. In general, if a 2-handle $\delta$ cancels a 1-handle $d$, and if other 2-handles $\delta_1, \delta_2, \ldots$ go over $d$, then we slide $\delta_i$ off $d$ using $\delta$. A typical case of this is shown in Fig. 6, where we have drawn the actual one-handle (usually this one-handle is only imagined).

To cancel $a_*$ and $\gamma$, we push off 6 copies of $\gamma$ and slide $\beta_1$, $\beta_2$ and $\beta_3$ off $a_*$. Erasing $a_*$ and $\gamma$, we get Fig. 7.

We redraw Fig. 7, simplifying the attaching maps (and deleting the boundary of the cube, leaving only the attaching curves). This gives Fig. 8, except for some framings which need explanation. Until now, the choice of a framing for a normal tube has been the obvious one. But in passing from Fig. 7 to Fig. 8, some twisting occurred as illustrated in Fig. 9. Plus (minus) one means one full right (left) handed twist in the framing.

In Fig. 8, the short arcs at the top and right can be pushed onto the boundary of the attaching 3-balls of $a_3$ and $a_2$, around those 1-handles, and then off the attaching 3-balls at the bottom and left, as in Fig. 10.

To get $\beta_2$ to cancel $a_3$, we must slide it over $\beta_1$. Push off a copy of $\beta_1$ using the $-1$
Fig. 7.

Fig. 8.

Thus becomes

Fig. 9.
framing, and band connect sum, as in Fig. 11. We redraw to get Fig. 12 (the +1 framing on $\beta_2$ is changed to 0 in the redrawing).

Now we cancel $a_1$, $a_2$, $a_3$ and $\beta_1$, $\beta_2 - \beta_1$, $\beta_3$ simultaneously, in analogy with Fig. 6. The result is Fig. 13, where the 1 and $-1$ denote full right and left-handed twists. Figure 13 is the unlink! Get 3 colors of chalk and a large blackboard; have fun.
Remarks. (1) We have not seriously tried to make this argument work for other examples of Cappell and Shaneson. Other matrices have more non-zero entries, making the attaching maps more complicated.

(2) It is clear from the theorem that there is a knotted 2-sphere $K$ in $S^4$ which is fibered by the punctured 3-torus with monodromy $A^2$. The exotic involution on $S^4$ restricts to the antipodal map on $K^2 \times B^2$ and to the map $(x, t) \rightarrow (Ax, t + \pi)$ on $T_0^3 \times S^1 = \mathcal{C}$. It would be a nontrivial exercise to draw $K$.

(3) Conjecture (Gluck): If a tubular neighborhood of a knot is removed from $S^4$ and sewn back in by the nontrivial diffeomorphism of $S^1 \times S^3$ coming from $\pi_1(SO(3))$, then the result is $S^4$.

If this operation is performed on $K$, then the resulting picture differs in that the 2-handle $\gamma$ is attached with a framing having one full twist. After cancelling $a_s$ with $\gamma$, we get Fig. 7 except with a full twist in the six lines parallel to the vector $(1/2, 1/2, 1)$. We have not been able to show that this manifold is $S^4$; indeed, it may be a counterexample to the conjectures of Gluck and Poincaré.

REFERENCES


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