ON THE STRENGTH OF MARRIAGE THEOREMS AND UNIFORMITY

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Abstract. Kierstead showed that every computable marriage problem has a computable matching under the assumption of computable expanding Hall condition and computable local finiteness for boys and girls. The strength of the marriage theorem reaches WKL or ACA if computable expanding Hall condition or computable local finiteness for girls is weakened. In contrast, the provability of the marriage theorem is maintained in RCA even if local finiteness for boys is completely removed. Using these conditions, we classify the strength of variants of marriage theorems in the context of reverse mathematics. Furthermore, we introduce another condition that also makes the marriage theorem provable in RCA, and investigate the sequential and Weihrauch strength of marriage theorems under that condition.

1. Introduction

1.1. Summary. A subset $R$ of the product $B \times G$ of two sets can be thought of as a multi-valued function (a set-valued function, or a bipartite graph), and written as $R : B \rightarrow G$. Given a multi-valued function $R$ between countable sets $B$ and $G$, we discuss whether it has a single-valued injective selection or not. Such a problem is called a marriage problem. Hall [11] showed that the marriage problem for a multi-valued function $R : B \rightarrow G$ is true whenever $R$ is locally finite (i.e., $R(b)$ has at most finitely many values for every $b \in B$) and fulfills the Hall condition (i.e., the cardinality of $R[X]$ is not less than that of $X$ for every finite set $X \subseteq B$). However, in the early age of recursive graph theory (cf. [9]), Manaster and Rosenstein [18] found that such a multi-valued function need not have a single-valued computable injective selection, even if it has a computable graph and its local finiteness is computably confirmed. To render the marriage theorem computable, Kierstead [17] introduced the notion of expanding Hall condition, which indicates that if the difference between $|R[X]|$ and $|X|$ tends to infinity as $|X|$ tends to infinity, where $X$ ranges over all finite subsets of $B$. Then, he found that $R$ has a single-valued computable injective selection whenever the graph of $R$ is computable, the local finiteness of $R$ is computably confirmed and the expanding Hall condition for $R$ is computably witnessed.

Concepts such as local finiteness and the Hall condition can be thought of as width conditions for $\Pi^0_1$ classes, because the set of all injective selections of a multi-valued function forms a $\Pi^0_1$ class in $\mathbb{N}^B \approx \omega^\omega$ ([4]). For instance, the local finiteness is known simply as compactness, and its computable version is known as recursive boundedness. Based on this observation, in the context of reverse mathematics, Hirst [13] (see also [12]) showed that the finite marriage theorem is provable in RCA, and that the infinite marriage theorem is equivalent to ACA (equivalently, König’s lemma) over RCA. Moreover, he showed that the infinite marriage theorem under the assumption of computable local finiteness is equivalent to WKL (König’s lemma for binary trees) over ACA.

Our aim is to clarify the relationship between such width conditions for $\Pi^0_1$ classes (problems) and the complexity of elements contained in them (solutions to them) in the context of reverse mathematics. To achieve this, in section 2 we investigate the strength of 24 variations (except 3 false variations) of marriage theorems obtained by combining three smallness conditions (namely, no local finiteness ($B, G$), local finiteness ($B', G'$), and highly recursiveness ($B'', G''$)) and three largeness conditions (namely, Hall condition ($H$), expanding Hall condition ($H'$), and computable expanding Hall condition ($H''$)).

The computable expanding Hall condition ($H''$) guarantees that every large number of inputs has sufficiently large number of outputs in order to make a computable marriage problem have a computable solution; however, there is another condition that implies this. In section 3, we introduce a new condition called constant bounded Hall condition ($H_b$), which requires that every finite set of inputs has few extra outputs. If a computable marriage problem fulfills this condition, then this problem will have a “non-uniformly” computable solution. In the practice of reverse mathematics, the sequential versions of $\Pi^0_1$ theorems, which expects to solve infinitely many instances of a particular problem simultaneously, have been investigated in order to reveal the necessity of the non-uniformity of their proofs in RCA. For instance, the intermediate value theorem is provable in ACA but its sequential version is equivalent to WKL [19]. We show in Section 3 that all of the marriage theorems with the constant bounded Hall condition ($H_b$) are “non-uniformly” provable in RCA, while some of their sequential versions are equivalent to WKL or ACA over ACA.

The sequential strength roughly suggests the non-uniformity level of computable principles. However, even if a $\Pi^0_2$ theorem $\tau$ is provable in RCA and its sequential version Seq($\tau$) is equivalent to WKL, it is considerably short of
determining the exact computational strength of \( f \). In Section 4, we employ the notion of Weihrauch reducibility to further analyze the computational strength of marriage theorems. Our investigation demonstrates the close connection between sequential reverse mathematics [14, 7, 6] and Brattka-Gherardi style reverse mathematics [2, 3] via Weihrauch reducibility.

The reader is referred to Simpson’s book [19] for basic knowledge of reverse mathematics including techniques for encoding mathematical statements in second order arithmetic. See [10] for the basic discussions on cardinality in weak first order arithmetic and the first order hierarchy.

1.2. Basic Terminology and Notation. Throughout this paper, for sets \( B \) and \( G \), we often identify each bipartite graph \( R(B, G) \) with a multi-valued function \( R \subseteq B \times G \) defined by \( R(b) = \{ g \in G : (b, g) \in R \} \). In addition, when the underlying sets \( B \) and \( G \) of a bipartite graph \( R(B, G) \) are clear from the context, we drop \( B, G \) and denote the bipartite graph just as \( R \) to avoid the notational complexity. In the graph theoretic terminology, each element of \( B \) and \( G \) is called “vertex”, and each element \((b, g)\) of \( R \) is called “edge”. One can think of \( B \) and \( G \) as the set of boys and girls, respectively. Then, \((b, g) \in R \) is regarded as that boy \( b \) knowing girl \( g \).

For a function \( f : B \rightarrow G \) and sets \( X \subseteq B \) and \( Y \subseteq G \), let \( f[X] \) and \( f^{-1}[Y] \) be the image of \( X \) and the preimage of \( Y \) under \( f \), and \( \text{dom}(f) \) and \( \text{rng}(f) \) be the domain and the range of \( f \). We also use the same notation for a multi-valued function \( R \subseteq B \times G \). That is, \( R[X] \) and \( R^{-1}[Y] \) denote \( \{ g \in G : (\exists b \in X) \ g \in R(b) \} \) and \( \{ b \in B : (\exists g \in Y) \ g \in R(b) \} \) respectively. In addition, by \( \text{dom}(R) \) and \( \text{rng}(R) \) we mean \( \{ b \in B : R(b) \neq \emptyset \} \) and \( R[B] \) respectively. Given \( Z \subseteq R \), by \( R - Z \) we denote the bipartite graph (multi-valued function) with \( \text{dom}(R - Z) = \text{dom}(R) \setminus \text{dom}(Z) \) and \( (R - Z)(b) := R(b) \setminus \text{rng}(Z) \) for every \( b \in \text{dom}(R - Z) \). Given \( V \subseteq B \cup G \), by \( R - V \) we denote the bipartite graph with \( \text{dom}(R - V) = \text{dom}(R) \setminus (V \cap B) \) defined by \( (R - V)(b) = R(b) \setminus (V \cap G) \) for every \( b \in \text{dom}(R - V) \). We denote a sequence by \( (.) \). For a \((\text{code of sequence }) s, \text{lh}(s) \) denotes the length of \( s \), \( s_i \) denotes the \( i \)-th element of \( s \) for \( i < \text{lh}(s) \), and \( s^*(t) \) denotes the concatenation of \( s \) and \( t \). \( C^* \) is the set of all finite sequences of the elements of a set \( C \). For a given set \( B \), \( X \subseteq B \) denotes that \( X \) is a \((\text{code of} )\) finite subset of \( B \). Let \( S_{\text{lh}}(X) := |R[X]| - |X| \) for \( X \subseteq B \).

We recall that \( \text{RCA}_0 \) consists of basic axioms for arithmetic, the \( \Sigma^0_1 \) induction scheme and the \( \Delta^0_1 \) comprehension scheme, \( \text{WKL}_2 \) consists of \( \text{RCA}_0 \) and \( \text{WKL} \) (weak König’s lemma), and \( \text{ACA}_0 \) consists of \( \text{RCA}_0 \) and \( \text{ACA} \) (the arithmetical comprehension scheme). (See [19] for the formal definitions.) In addition, \( \Sigma^0_n \)-IND denotes the \( \Sigma^0_n \) induction scheme and \( \text{RCA} \) denotes the extension of \( \text{RCA}_0 \) with the full second order induction scheme.

1.3. Main Results. First, we state the precise definition of each notion in \( \text{RCA}_0 \). A bipartite graph \( R(B, G) \) is \( B \)-locally finite if \(|R[b]| < \infty \) for all \( b \in B \). The graph is \( B \)-highly recursive or computably \( B \)-locally finite if there is a function \( f : B \rightarrow \mathbb{N} \) such that \( f(b) = |R[b]| \) for all \( b \in B \). The notion of being \( G \)-local finiteness and \( G \)-highly recursiveness are defined in the same manner. The graph \( R(B, G) \) satisfies the Hall condition if \( |R[X]| \geq |X| \) holds for all \( X \subseteq B \); it satisfies the expanding Hall condition if it satisfies the Hall condition, and, for every \( n \in \mathbb{N} \), there is \( m \in \mathbb{N} \) such that the difference \( S_{\text{lh}}(X) = |R[X]| - |X| \) is not less than \( n \) for all \( X \subseteq B \) such that \( |X| \geq m \). If there is a function in \( \text{RCA}_0 \) mapping each \( n \) to such an \( m \), then we say that \( R(B,G) \) satisfies the computable expanding Hall condition. A solution, a matching or an injective selection of \( R(B,G) \) is an injective single-valued function \( M \subseteq R \), i.e., an injection \( M : B \rightarrow G \) with \( M(b) \in R(b) \). Hereafter, we use the following notation:

- \( X \) : no local finiteness for \( X \), for \( X \in \{ B, G \} \).
- \( X' \) : \( X \)-locally finite, for \( X \in \{ B, G \} \).
- \( X'' \) : \( X \)-highly recursive, for \( X \in \{ B, G \} \).
- \( H \) : Hall condition.
- \( H' \) : expanding Hall condition.
- \( H'' \) : computable expanding Hall condition.

**Statement** \( \text{((B''_M G''_M), (G''_M), (H''_M))} \). If a bipartite graph \( R(B,G) \) satisfies \( B(j), G(j), \) and \( H(j) \), then \( R(B,G) \) has a solution.

We investigate the strength of all marriage theorems having the above form. Using this terminology, we can rephrase the result of Hirst [13] as follows: The statement \( B''_M G''_M \) is equivalent to \( \text{ACA} \), and \( B''_M G''_M \) is equivalent to \( \text{WKL} \) over \( \text{RCA}_0 \). Furthermore, Kierstead [17] showed that \( B''_M G''_M \) holds effectively. Now we note that the Hall condition has \( \Pi^0_2 \) form but it can be written as a \( \Pi^0_2 \) formula under the assumption of being \( B \)-highly recursive. Thus, we can verify Kierstead’s proof in our base system \( \text{RCA}_0 \) with the \( \Sigma^0_1 \) induction scheme (which enable us to carry out \( \Pi^0_2 \) reduction [19, Corollary II.3.10]). Hence, \( B''_M G''_M \) is provable in \( \text{RCA}_0 \). Furthermore, the sequential version of \( B''_M G''_M \) is also provable in \( \text{RCA}_0 \) by imitating the proof.

Our results in Section 2 are summarized in Table 1. Consequently, we find that the two conditions \( G'' \) and \( H'' \) are necessary and sufficient for a computable marriage problem to have a computable solution. Except in these cases, \( G'' \) and \( H'' \) do not affect the strength of marriage theorems, and only condition \( B'' \) is necessary and sufficient for a marriage theorem to be provable in \( \text{WKL}_2 \).
RCA$_0$ injections with pairwise disjoint ranges. The basic idea of the proof is to construct infinitely many disjoint marriage

Hirst [13, Theorem 2′′]

We extend Kierstead’s proof of [17, Theorem 5] to show this lemma. We reason in RCA$_0$ + $\Sigma^0_3$. We construct a set $B;G$ as follows (see also Fig. 1). At the first step, we in advance

Table 1. The strength of marriage theorems ($\ast$ : false)

\begin{tabular}{|c|c|c|}
\hline
Hall condition & Expanding Hall condition & Recursive expanding Hall condition \\
\hline
AC$_A_0$ & $\ast$ & $B_G\n$-M & $B_G\n$-M & $B_G\n$-M & $\neg$ RCA$_0$ + $\Sigma^0_3$ \\
\hline
AC$_A_0$ & $\ast$ & $B_G\n$-M & $B_G\n$-M \\
\hline
AC$_A_0$ & $\ast$ & $B_G\n$-M & $B_G\n$-M & $\neg$ RCA$_0$ + $\Sigma^0_3$ \\
\hline
B$_{G-M}$ [13] & $B_G\n$-M & $B_G\n$-M & $B_G\n$-M \\
\hline
B$_{G-M}$ & $B_G\n$-M & $B_G\n$-M & $B_G\n$-M \\
\hline
B$_{G-M}$ & $B_G\n$-M & $B_G\n$-M & $B_G\n$-M & $\neg$ RCA$_0$ + $\Sigma^0_3$ \\
\hline
WKL$_0$ & $B_G\n$-M [13] & $B_G\n$-M & $B_G\n$-M \\
\hline
WKL$_0$ & $B_G\n$-M & $B_G\n$-M & $B_G\n$-M \\
\hline
RCA$_0$ & $B_G\n$-M & $B_G\n$-M & $B_G\n$-M & $\neg$ RCA$_0$ ([17]) \\
\hline
\end{tabular}

Table 2. The sequential strength of constant bounded marriage theorems, which are provable in RCA$_0$

On the other hand, already mentioned in Subsection 1.1, there is another approach to make a computable marriage problem have a computable solution. As an extreme case, if the Hall condition holds and each boy knows at most one girl, then it obviously has a computable solution. Based on this observation, we introduce another kind of Hall condition, which requires that boys have few extra acquaintances. A bipartite graph $R(B,G)$ satisfies the constant bounded Hall condition if there exists $k$ such that for all $X \subseteq B$, $|X| \leq |R[X]| \leq |X| + k$ holds. We use the symbol $H_{cb}$ for the constant bounded Hall condition. Our results in Section 3 are summarized in Table 2. Summarizing, a marriage theorem with condition $H_{cb}$ is uniformly computable if and only if it includes $G''$.

2. Marriage Theorems with Expanding and Recursive Expanding Hall Condition

2.1. Reversals. Kierstead [17, Theorem 5] showed that $B_G\n$-M does not hold in the least $\omega$-model of RCA$_0$, while Hirst [13, Theorem 2.3] showed that it is provable in WKL$_0$. The next lemma means that it is actually equivalent to WKL over RCA$_0$.

Lemma 2.1. RCA$_0$ $\vdash B_G\n$-M $\rightarrow$ WKL, that is, the following assertion implies WKL over RCA$_0$: If $R(B,G)$ is a bipartite graph which is $B,G$-highly recursive and satisfies the expanding Hall condition, then $R(B,G)$ has a solution.

Proof. We extend Kierstead’s proof of [17, Theorem 5] to show this lemma. We reason in RCA$_0$. It suffices to separate the range of disjoint injections, which is equivalent to WKL over RCA$_0$ ([19, Lemma IV.4.4]). Let $f,g : \mathbb{N} \rightarrow \mathbb{N}$ be given injections with pairwise disjoint ranges. The basic idea of the proof is to construct infinitely many disjoint marriage problems such that the solution of the $i$-th problem indicates whether $i$ is in $\text{rng}(f)$ or $\text{rng}(g)$. That is, the bipartite graph $R(B,G)$ produced eventually is the disjoint union of the bipartite graphs $R_i(B_i,G_i)$, $i \in \mathbb{N}$. Here we describe the construction of the $i$-th graph $R_i(B_i,G_i)$. The underlying set of the graph $R_i(B_i,G_i)$ is the disjoint union of $3(i+1)$ many infinite full binary trees $\{0,1\}^*$. Thus, each $v \in B_i \cup G_i$ is described as $(k,\sigma) \in T_i = 3(i+1) \times \{0,1\}^*$. The sets of the boys $B_i$ and the girls $G_i$ in the graph $R_i(B_i,G_i)$ are chosen as $B_i = \{(k,\sigma) \in T_i : lh(\sigma) \text{ is even}\}$ and $G_i = \{(k,\sigma) \in T_i : lh(\sigma) \text{ is odd}\}$.

We construct a set of edges on the graph $R_i(B_i,G_i)$ as follows (see also Fig. 1). At the first step, we in advance connect each boy in the 0-th column with his two successor girls in the 1-st column. Now we consider three cases for the construction of $R_i(B_i,G_i)$ in the $(2j+2)$-th column. Let $U^j_1$ be the set of the first $(i+1)2^{j+1}$ boys in the $(2j+2)$-th column, and let $G^L_j$ (resp. $G^R_j$) be the set of all $(k,\sigma) \in T_i$ with $\sigma(0) = 0$ (resp. $\sigma(0) = 1$) in the $(2j+1)$-th column.

(1) If neither $f(j) = i$ nor $g(j) = i$ we connect each boy in the $(2j+2)$-th column with the predecessor girl in the $(2j+1)$-th column and the two successor girls in the $(2j+3)$-th column respectively.

(2) If $f(j) = i$, then we connect $U^j_1$ with $G^L_j$ completely and each remaining boy in the $(2j+2)$-th column with the two successor girls in the $(2j+3)$-th column respectively.

(3) If $g(j) = i$, then we connect $U^j_1$ with $G^R_j$ completely and each remaining boy in the $(2j+2)$-th column with the two successor girls in the $(2j+3)$-th column respectively.

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Importantly, it inductively follows from this construction that for all \( l \leq j \), at least the two-third of boys in the left (resp. right) sides in \( 2l \)-th column (see Fig. 1) choose their predecessor girls whenever \( f(j) = i \) (resp. \( g(j) = i \)).

It is trivial that \( R(B, G) \) is \( B, G \)-highly recursive. We show that \( R(B, G) \) satisfies the expanding Hall condition.

**Claim 2.2 (\( \text{RCA}_0 \)).** If \( i \) is contained in \( \text{rng}(f) \cup \text{rng}(g) \), then either \( S_{R_i}(X) \geq |X| \) or \( S_{R_i}(X) \geq i+1 \) holds for \( X \subset B_i \).

(Proof of Claim.) Let \( i \) be contained in \( \text{rng}(f) \cup \text{rng}(g) \) at \( j \), and fix a finite subset \( X \) of \( B_i \). If \( X \) and \( U_i^j \) are disjoint, then \( S_{R_i}(X) \geq |X| \) holds since each boy in \( X \) is connected with the two successor girls. Assume that \( X \) intersects \( U_i^j \). Consider the following set.

\[
V_i^j = \begin{cases} \{(k, \sigma) \in B_i : 0 < lh(\sigma) \leq 2j, \text{ and } \sigma(0) = 0\} & \text{if } i \in \text{rng}(f), \\ \{(k, \sigma) \in B_i : 0 < lh(\sigma) \leq 2j, \text{ and } \sigma(0) = 1\} & \text{if } i \in \text{rng}(g). \end{cases}
\]

Note that the boys in the 0-th column are not contained in \( V_i^j \). We separate \( R_i(B_i, G_i) \) into two subgraphs. Let \( W_i := U_i^j \cup V_i^j \). Then \( S_{R_i-W_i}(X \setminus W_i) \geq 0 \) since each boy in \( X \setminus W_i \) is connected with at least one successor girl. To estimate the value \( S_{R_i}(X \cap W_i) \), we first note that this value is equal to \( |R_i[X \cap W_i]| - |X \cap V_i^j| - |X \cap U_i^j| \). Moreover, \( |R_i[X \cap W_i]| \geq |R_i[U_i^j]| + 2^{-1}|X \cap V_i^j| \) holds since each boy in \( U_i^j \) knows all girls in \( R_i[U_i^j] \), and, for each boy in \( V_i^j \), his predecessor girl has just two successor boys. By our definition, \( |R_i[U_i^j]| = |G^{R_i(j)}_k| = |G^{R_i(j)}_k| = 3(i + 1)2^{2j}, \) \( |U_i^j| = (i + 1)2^{2j + 1} \), and

\[
|V_i^j| = 3(i + 1) \sum_{l=0}^{j-1} 2^{2l+1} = 2(i + 1)(2^{2j} - 1).
\]

Therefore,

\[
S_{R_i}(X \cap W_i) \geq |R_i[U_i^j]| - \frac{1}{2} |X \cap V_i^j| - |X \cap U_i^j| \\
\geq 3(i + 1)2^{2j} - (i + 1)(2^{2j} - 1) - (i + 1)2^{2j + 1} \\
= i + 1.
\]

Hence, we have \( S_{R_i}(X) \geq S_{R_i-W_i}(X \setminus W_i) + S_{R_i}(X \cap W_i) \geq i + 1. \) \( \Box \)

Let \( W_{<n} \) be the union of the sets \( W_i \) such that \( i < n \) and \( i \in \text{rng}(f) \cup \text{rng}(g) \). If \( i \) is not contained in \( \text{rng}(f) \cup \text{rng}(g) \), then \( S_{R_i}(X) \geq |X| \) for \( X \subset B_i \), since the graph structure of \( R_i(B_i, G_i) \) also corresponds exactly to the disjoint union of \( 3(i + 1) \) many full binary trees. By this fact and Claim 2.2, it is not hard to see that for all \( n \), if a finite subset \( X \) of \( B \) contains at least \( n + |W_{<n}| \) elements, then \( S_R(X) \geq n \) holds (see the proof of [17, Theorem 5] for details). Thus, \( R(B, G) \) satisfies the expanding Hall condition.

Consequently, the assertion \( \text{B}_j^{\text{M}} \rightarrow \text{M} \) ensures that \( R(B, G) \) has a solution \( M \). Let \( M_{b,0} \) be the set of all boys \( b \) in the 0-th column of \( R_i(B_i, G_i) \) who chooses the left successor girl according to \( M \) (i.e., \( M(b) = (k, 0)) \) for \( k < 3(i + 1) \). By \( \Sigma_0^0 \) comprehension, the set \( S = \{ i \in \mathbb{N} : |M_{b,0}| \leq i + 1 \} \) exists. We shall show that \( S \) separates the
ranges of $f$ and $g$. Suppose $i \in \text{rng}(f)$ and $i \notin S$. Then $|M_{i,0}| > i + 1$. Now
\[ |U_i^j \cup V_i^j| = |U_i^j| + |V_i^j| = (i + 1)2^{2j+1} + 2(i + 1)(2^{2j} - 1) = (i + 1)2^{2j+2} - 2(i + 1). \]
However,
\[ |M[U_i^j \cup V_i^j]| = 3(i + 1) \sum_{l=0}^{j} 2^{2l} - |M_{i,0}| < (i + 1)2^{2j+2} - 2(i + 1). \]
This contradicts the injectivity of $M$. Thus $i \in \text{rng}(f)$ implies $i \in S$. In the same manner, we can show that $i \in \text{rng}(g)$ implies $i \notin S$. □

By extending the previous proof, we can show the next lemma.

**Lemma 2.3.** $\text{RCA}_0 \vdash \text{B}_{0}^{\omega}G'' - M \rightarrow \text{ACA}$. That is, the following assertion implies ACA over $\text{RCA}_0$: If $R(B, G)$ is a bipartite graph which is $B$-locally finite, $G$-highly recursive and satisfies the expanding Hall condition, then $R(B, G)$ has a solution.

**Proof.** We reason in $\text{RCA}_0$. It suffices to find the range of an injection, which is equivalent to ACA over $\text{RCA}_0$ ([19, Lemma III.1.3]). Let $f : \mathbb{N} \to \mathbb{N}$ be a given injection. We can show the existence of range of $f$ in a slightly little different way from Lemma 2.1. As in the previous proof, we construct infinitely many disjoint marriage problems such that the solution of $i$-th problem indicates whether $i$ is in $\text{rng}(f)$ or not. The construction of $R_i(B_i, G_i)$ is similar to the previous one. The underlying set of $R_i(B_i, G_i)$ consists of the disjoint union of $(3(i + 1)$ infinite binary trees pruned below the right girls in the 1-st column and with additional $2(i + 1)$ boys. Formally, the underlying set $T_i = B_i \cup G_i$ of $R_i(B_i, G_i)$ is defined as $T_i = \{(k, \sigma) \in 3(i + 1) \times \{0, 1\}^* : \sigma(0) \neq 1 \} \cup J_i$, where $J_i$ is a set of $2(i + 1)$ vertices (see also Fig. 2), and put $B_i = \{(k, \sigma) \in T_i : lh(\sigma) = \text{even} \} \cup J_i$ and $G_i = \{(k, \sigma) \in T_i : lh(\sigma) = \text{odd} \}$. We construct $R_i$ as follows. In advance, we connect each boy in the 0-th column with his two successor girls in the 1-st column as before. Moreover, connect each right girl in the 1-st column with the exceptional boys in $J_i$ completely. Note that the number of the right girls in the 1-st column is $3(i + 1)$ and $|J_i| = 2(i + 1)$. Then we determine who are connected with the boys in the $(2j + 2)$-th column according to $f$ as follows. (See also Fig. 2.)

1. If $f(j) \neq i$, we connect each boy in the $(2j + 2)$-th column with the girls in the $(2j + 1)$-th column and $(2j + 3)$-th column as before.
2. If $f(j) = i$, we not only connect $U_i^j$ with $G_i^{L(i)}$ completely and each remaining boy in the $(2j + 2)$-th column with the two successor girls in the $(2j + 3)$-th column respectively as before, but also connect each boy in $J_i$ with some two girls remaining in the $(2j + 3)$-th column disjointly.

In the above construction, the procedure that combining $J_i$ with the girls remaining in the $(2j + 3)$-th column has the role of “liberating” the right girls in the 1-st column from the proposal by the boys in $J_i$. Since we use this way of revising several times in the proofs below, we shall call this technique “liberation method”.

Obviously the graph $R(B, G)$ produced in this way is $B$-locally finite and $G$-highly recursive and we can show that $R(B, G)$ satisfies the expanding Hall condition as in Lemma 2.1. Then, the assertion $\text{B}_{0}^{\omega}G'' - M$ ensures that $R(B, G)$ has a solution $M$. As in the previous proof, let $S$ be the set of all numbers $i$ with $|M_{i,0}| \leq i + 1$, where $M_{i,0}$ is the set of boys in the 0-th column of $R_i(B_i, G_i)$ who choose their left successor girls according to $M$. As before, it is easy to see that $i \in \text{rng}(f)$ implies $i \in S$. If $i \notin \text{rng}(f)$, then the $2(i + 1)$ boys in $J_i$ just know the right girls in the 1-st
column (i.e., the girls of form \((k, 1)\) \(\in T_i\)). Therefore, \(2(i + 1)\) right girls in the 1-st column must be chosen by boys in \(J_i\). Then, at most \(i + 1\) boys in the 0-th column can choose their right girls. Thus, \(|M_{i, 0}| > i + 1\), and so \(i \not\in S\). Consequently, \(S = \text{rng}(f)\).

**Lemma 2.4.** RCA\(_0\) \(\vdash B^m_{\text{loc}}, G^*\cdot M \rightarrow \text{WKL}\), that is, the following assertion implies WKL over RCA\(_0\): If \(R(B, G)\) is a bipartite graph which is \(B\)-highly recursive, \(G\)-locally finite, and satisfies the computable expanding Hall condition, then \(R(B, G)\) has a solution.

**Proof.** We reason in RCA\(_0\). It suffices to separate the range of disjoint functions ([19, Lemma IV.4.4]). Let \(f, g : \mathbb{N} \rightarrow \mathbb{N}\) be given injections with pairwise disjoint ranges. We construct a bipartite graph \(R(B, G)\) which is the disjoint union of bipartite graphs \(R_i(B_i, G_i), i \in \mathbb{N}\). Put \(B_i = G_i = \mathbb{N}\). For convenience, we will suppress the coding and label the \(j\)-th boy in \(B_i\) by \(b^i_j\) as well as the \(2j\)-th and \((2j + 1)\)-th girls in \(G_i\), by \(g^i_{(1)}\) and \(g^i_{(2)}\) respectively. Then we construct \(R_i\) as follows (see also Fig. 3).

1. The pairs \((b^i_u, g^i_{(u)})\) and \((b^i_v, g^i_{(v)})\) are enumerated into \(R_i\) for each \(u \in \{0, 1, 2\}\).
2. If neither \(f(j) = i\) nor \(g(j) = i\) holds, then the pairs \((b^i_{2j+3+v}, g^i_{(2j+3+v)})\) and \((b^i_{2j+3+v}, g^i_{(2j+3+v)})\) are enumerated into \(R_i\) for each \(v \in \{0, 1\}\).
3. If \(f(j) = i\) holds, then \((b^i_{2j+3+v}, g^i_{(u)})\) is enumerated into \(R_i\) for each \(u \in \{0, 1, 2\}\) and \(v \in \{0, 1\}\).
4. If \(g(j) = i\) holds, then \((b^i_{2j+3+v}, g^i_{(u)})\) is enumerated into \(R_i\) for each \(u \in \{0, 1, 2\}\) and \(v \in \{0, 1\}\).

It is trivial that \(R(B, G)\) is \(B\)-highly recursive and \(G\)-locally finite. We show that \(R(B, G)\) satisfies the computable expanding Hall condition.

**Claim 2.5 (RCA\(_0\)).** For a finite subset \(X\) of \(B_i\), \(|X| \geq 5n\) implies \(S_R(X) \geq n\).

**(Proof of Claim.)** Let \(X\) be a finite subset of \(B\) such that \(|X| \geq 5n\). For each \(i\), let \(B^i_j\) be the set of all boys of the form \(b^i_j\) such that \((b^i_j, g^i_{(0)})\) or \((b^i_j, g^i_{(1)})\) is enumerated into \(R_i\). Since the range of \(f\) and \(g\) are disjoint, \(S_R(X \cap B^i_j) \geq S_R(X \cap B^i_j)\) for each \(i \in \mathbb{N}\). Then \(S_R(X) = \sum_i S_R(X \cap B^i_j) \geq \sum_i S_R(X \cap B^i_j)\). Note that, if \(X \cap B^i_j \neq \emptyset\), then \(S_R(X \cap B^i_j) \geq 1\). Therefore, in the case of \(|\{i : X \cap B^i_j \neq \emptyset\}| \geq n\), we have \(S_R(X) \geq n\). In the case of \(|\{i : X \cap B^i_j \neq \emptyset\}| \leq n\), there are at least \(3n\) boys who are not in \(\bigcup B^i_j\). Since each of these boys knows under 2 girls (see Fig. 3), \(|R(X)| \geq 6n\), so \(S_R(X) = |R(X)| - |X| \geq 6n - 5n = n\). \(\square\)

By the previous claim, \(R(B, G)\) satisfies the computable expanding Hall condition.

Then there exists a solution \(M\) of \(R(B, G)\) by \(B^\text{loc}_{\text{loc}}, G^\text{loc}\cdot M\). By \(\Delta^0_1\) comprehension, take

\[V := \{i \mid \text{two of } \{b^i_0, b^i_1, b^i_2\} \text{ choose their right girls via } M\}.\]

If \(i \in \text{rng}(f)\), then \(b^i_{2j+3}\) and \(b^i_{2j+4}\) choose two of \(\{g^i_{(0)}, g^i_{(1)}, g^i_{(2)}\}\) via \(M\), then two of \(\{b^i_0, b^i_1, b^i_2\}\) must choose their right girls. Hence, \(i \in V\). If \(i \in \text{rng}(g)\), then \(b^i_{2j+3}\) and \(b^i_{2j+4}\) choose two of \(\{g^i_{(0)}, g^i_{(1)}, g^i_{(2)}\}\) via \(M\). Hence, \(i \not\in V\). \(\square\)

In the context of recursive graph theory, Lemma 2.4 suggests that being \(G\)-highly recursive is essential for a computable marriage problem with the computable expanding Hall condition to have a computable solution.

By applying “liberation method” to the proof of Lemma 2.4, we can show the next lemma.
Lemma 2.6. \( \text{RCA}_0 \vdash \text{B}^0_{1\text{ir}} \cdot \text{G}^* \cdot \text{M} \rightarrow \text{ACA} \), that is, the following assertion implies ACA over \( \text{RCA}_2 \): If \( R(B,G) \) is a bipartite graph which is \( B,G \)-locally finite and satisfies the computable expanding Hall condition, then \( R(B,G) \) has a solution.

Proof. We reason in \( \text{RCA}_0 \). It suffices to find the range of an injection \( f : \mathbb{N} \rightarrow \mathbb{N} \) ([19, Lemma III.1.3]). The construction of \( R \) is similar to that in Lemma 2.4. But in this occasion, for each \( i \)-th graph \( R_i(B,G) \), we make the top right three girls being connected by two exceptional boys \( b'_i, b''_i \) in advance. Then if \( f(i) = i \), we carry out the following procedure in the \( i \)-th graph \( R_i(B,G) \).

1. Combine \( b'_{i+3}, b'_{i+4} \) with the top left girls \( g^1_i, g^1_i, g^2_i \) completely.
2. Combine \( b''_i \) with \( g^1_i, g^2_i, g^2_i \) and \( b''_i \) with \( g^1_i, g^1_i, g^2_i \).

The procedure 2 has the role of “liberating” the top right three girls from the proposal by \( b'_i, b''_i \). Then one can see that \( R(B,G) \) is \( B,G \)-locally finite and satisfies the computable expanding Hall condition by taking \( h : \mathbb{N} \rightarrow \mathbb{N} \) such that \( h(n) = 5n \) as in Lemma 2.4. \( \text{B}^0_{1\text{ir}} \cdot \text{G}^* \cdot \text{M} \) ensures that \( R(B,G) \) has a solution \( M \) and \( V := \{ i \mid \text{two of } \{ b'_{i}, b''_{i}, b''_{i} \} \text{ choose their right girls via } M \} \) is the range of \( f \) as before.

\( \square \)

2.2. Proofs in \( \text{RCA}_0 + \Sigma^0_3 \text{-IND} \). We recall that a computable marriage problem has a computable solution under the three strongest assumptions \( B'' \), \( G'' \), and \( H'' \) as shown in [17]. As we have seen in Lemmas 2.1 and 2.4, in general, it has no computable solution under the absence of \( G'' \) or \( H'' \). What will happen when \( B'' \) is weakened? Surprisingly, even in the absence of \( B' \), every computable marriage problem has a computable solution when it satisfies \( G'' \) and \( H'' \).

Theorem 2.7. \( \text{RCA}_0 + \Sigma^0_3 \text{-IND} \vdash \text{B}^0_{1\text{ir}} \cdot \text{G}^* \cdot \text{M} \), that is, the following is provable in \( \text{RCA}_0 + \Sigma^0_3 \text{-IND} \): If \( R(B,G) \) is a bipartite graph which is \( G \)-highly recursive and satisfies the computable expanding Hall condition, then \( R(B,G) \) has a solution.

We need the following lemmas to show the previous theorem.

Lemma 2.8 (\( \text{RCA}_0 + \Sigma^0_3 \text{-IND} \)). Assume that \( R(B,G) \) is a bipartite graph with the expanding Hall condition. Then for any \( b \in B \) there exists \( g \in R[b] \) such that the remaining graph \( R - \{(b,g)\} \) satisfies the Hall condition.

Proof. We consider in two cases. We first consider in the case that given \( b \) knows infinitely many girls. If \( |X| < |R[X]| \) for all \( X \subseteq B \), our requirement clearly holds. We assume that there exists \( X \subseteq B \) such that \( |X| \geq |R[X]| \), which implies \( |X| = |R[X]| \) by the Hall condition. Then there exists a maximal finite subset \( X_0 \subseteq X \) such that \( |X_0| \geq |R[X_0]| \), since if it is not, \( \Sigma^0_3 \text{-IND} \) induction can prove a contradiction to the expanding Hall condition. (Note that \( |X_0| \geq |R[X_0]| \) is written as a \( \Pi^0_1 \) formula.) Now \( b \notin X_0 \). By using \( \Pi^0_1 \) collection principle (provable in \( \text{RCA}_0 + \Sigma^0_3 \text{-IND} \)), \( R[X_0] \) is assumed to be finite. Therefore, \( b \) must know some \( g \notin R[X_0] \). We shall show that \( R - \{(b,g)\} \) satisfies the Hall condition. Suppose not. Then there exists nonempty \( X_1 \subseteq B \) such that \( |X_1| = |R[X_1]| \) and \( g \in R[X_1] \). So, \( |X_0 \cup X_1| = |R[X_0] \cup R[X_1]| \) follows from it. Moreover \( X_0 \cup X_1 \supseteq X_0 \) follows from \( g \notin R[X_0] \). This contradicts the maximality of \( X_0 \).

Secondly, we consider in the case that given \( b \) knows at most finitely many girls. Let \( B_f \) denote the set of all boys knowing at most finitely many girls. When \( B_f \) is finite, \( \text{RCA}_0 + \Sigma^0_3 \text{-IND} \) proves the existence of \( B_f \), since \( B_f \) is \( \Sigma^0_3 \) definable and \( \Sigma^0_3 \text{-IND} \) implies bounded \( \Sigma^0_3 \) comprehension (cf. [19, Exercise II.3.13]). Therefore, our requirement clearly holds because the finite marriage theorem is provable in \( \text{RCA}_0 \) ([13, Theorem 2.4]). Next we assume that \( B_f \) is infinite. For the sake of contradiction, suppose that there is no \( g \in R[b] \) such that \( G - (b,g) \) satisfies the Hall condition. Since the original graph \( R(B,G) \) satisfies the Hall condition, there is a finite set \( X_g \subseteq R^{-1}[g] \setminus \{ b \} \) such that \( |X_g| = |R[X_g]| \) for all \( g \in R[b] \). Put \( X_1 = \bigcup_{g \in R[b]} X_g \). Then, since \( R[b] \) is finite, \( \Sigma^0_3 \text{-IND} \) implies that \( X_1 \) is finite and \( |X_1| = |R[X_1]| \). Since \( b \notin X_1 \) and \( R[b] \subseteq R[X_1] \), we have

\[ |X \cup \{ b \}| = |R[X_1]| + 1 > |R[X_1]| = |R[X_1] \cup \{ b \}|. \]

This is the desired contradiction. Hence, there is \( g \in R[b] \) such that \( R - \{(b,g)\} \) satisfies the Hall condition.

\( \square \)

**Definition 2.9** (\( R \)-chain, chaining matched).

1. A finite sequence \( s = (s^B_j, s^G_j)_{j<k} \) is a \( R \)-chain with starting point \( b \) of length \( k \) (\( \geq 0 \)) if \( (s^B_j)_{j<k} \) and \( (s^G_j)_{j<k} \) are nondecreasing sequences of finite subsets of \( B \) and \( G \) respectively, where \( s^B_0 = \{ b \} \), \( s^G_j \subseteq R[s^B_j], s^B_{j+1} = R^{-1}[s^G_j] \), and \( R(s^B_j, s^G_j) \) satisfies the Hall condition for each \( j < k \). A \( R \)-chain \( (s^B_j, s^G_j)_{j<k} \) is called proper if \( (s^B_j)_{j<k} \) is strictly increasing.
2. A finite matching \( m \subseteq \mathbb{N} \times R \) is \( R \)-chainable with starting point \( b \) if there is a \( R \)-chain \( s = (s^B_j, s^G_j)_{j<k} \) with starting point \( b \) such that \( \text{dom}(m) = s^B_{k-1} \) and \( s^G_j = m[s^B_j] \) for all \( j < k \).

Note that every $R$-chainable matching $m$ with an $R$-chain $s$ has the following disjointness property:

\[(1) \quad R[B \setminus s_{j+1}^R] \cap m[s_j^R] = \emptyset,\]

for each $j < \ell h(s) - 1$. The next technical lemma has a key role in our proof of Theorem 2.7.

**Lemma 2.10 (RCA$_0 + \Sigma^0_3$-IND).** Assume that $R(B, G)$ is a bipartite graph which is $G$-highly recursive and satisfies the expanding Hall condition. Then, for all $b \in B$ and $l \in \mathbb{N}$, there exists an $R$-chainable matching with starting point $b$ which has an $R$-chain of length $l$ such that $R \backslash m$ satisfies the Hall condition.

**Proof.** Since $R(B, G)$ is $G$-highly recursive, there exists a function $q$ from codes of finite subset $Y$ of $G$ to that of $B$ such that $q(Y) = R^{-1}[Y]$. We fix $b \in B$ and show our lemma by induction on $l$. Since the Hall condition is a $\Pi^0_3$ formula, this is a $\Sigma^0_3$ induction and carried out straightforwardly by the iterative use of Lemma 2.8 and the function $q$ within RCA$_0 + \Sigma^0_3$-IND.

We are now prepared to show Theorem 2.7.

**Proof of Theorem 2.7.** We reason in RCA$_0 + \Sigma^0_3$-IND. It suffices to consider only the case that $B$ is infinite since the finite marriage theorem is provable in RCA$_0$ ([13, Theorem 2.1]). Let $\{b_i : i \in \mathbb{N}\}$ be an enumeration of $B$ and $h$ be a witness of the computable expanding Hall condition. We shall now construct a solution of $R(B, G)$ by a recursive procedure. Let $\theta(u, v)$ say that $u = \langle u_i : i \leq v \rangle$ is a sequence of length $v + 1$ and for all $i < v + 1$, $u_i$ is the least $R_i$-chainable matching with starting point $b_i$ which has an $R_i$-chain of length $h(i + 1) + 1$, where $R_i(B, G_i) := R_i\langle B, G \rangle - \langle \langle b_i, u_i; (b_i) \rangle : i < i \rangle$, namely, the remaining graph obtained by removing the previously determined matching pairs from $R(B, G)$. If $\theta(u, v)$ holds, then we identify each $R_i$-chainable matching $u_i$ with the $R_i$-chain $s_i$ for $b_i$.

We reason in RCA$_0 + \Sigma^0_3$-IND. It suffices to consider only the case that $B$ is infinite since the finite marriage theorem is provable in RCA$_0$ ([13, Theorem 2.1]). Let $\{b_i : i \in \mathbb{N}\}$ be an enumeration of $B$ and $h$ be a witness of the computable expanding Hall condition. We shall now construct a solution of $R(B, G)$ by a recursive procedure. Let $\theta(u, v)$ say that $u = \langle u_i : i \leq v \rangle$ is a sequence of length $v + 1$ and for all $i < v + 1$, $u_i$ is the least $R_i$-chainable matching with starting point $b_i$ which has an $R_i$-chain of length $h(i + 1) + 1$, where $R_i(B, G_i) := R_i\langle B, G \rangle - \langle \langle b_i, u_i; (b_i) \rangle : i < i \rangle$, namely, the remaining graph obtained by removing the previously determined matching pairs from $R(B, G)$. If $\theta(u, v)$ holds, then we identify each $R_i$-chainable matching $u_i$ with the $R_i$-chain $s_i$ for $b_i$.

Note that $\theta(u, v)$ is $\Sigma^0_3$, since $R(B, G)$ is $G$-highly recursive. We shall decide the partner of $b_i$ as the girl indicated via uniquely determined matching $u_i$. Suppose that we have shown $\forall v \exists u(\theta(u, v))$. Then the witness $u^v$ for each $v$ in $\mathbb{N}$ is a initial segment of $u^v$ for $v_2 \leq v$ because of the minimality of each $u_i$ in the definition of $\theta(u, v)$. Therefore by $\Delta^0_3$ comprehension (as primitive recursion in RCA$_0$, see [19, Theorem II.3.4]), there exists a function which outputs the unique $u^v$ for each $v \in \mathbb{N}$. Take $M : B \rightarrow G$ by $M(b_i) = (u^v)_i(b_i)$, then it is not hard to see that $M$ is an injection from $B$ to $G$. Thus our goal is to prove $\forall v \exists u(\theta(u, v))$. In preparation, we first show the following claim.

**Claim 2.11 (RCA$_0 + \Sigma^0_3$-IND).** For all $u$ and $v$, if $\theta(u, v)$ holds, then $R - \{b_j, u_j(b_j) : j < v + 1\}$ satisfies the Hall condition.

**(Proof of Claim.)** We fix $u$ and $v$ such that $\theta(u, v)$ holds and show that for all $i \leq v$, $R - \{b_j, u_j(b_j) : j < i + 1\}$ satisfies the Hall condition by induction on $i$. Since the Hall condition is $\Pi^0_3$, this is a $\Pi^0_3$ induction. We shall show only the initial step below. The induction step can be shown in the same manner by using the induction hypothesis. Let $R_1 := R - \{b_0, u_0(b_0)\}$, $s_0$ be an $R$-chain for $u_0$ and fix $X \subseteq B \setminus \{b_0\}$.

In the case that $s_0 = \langle s_{0, j} : j \leq k \rangle$ is non-proper. Let $\langle s_{0, j} : j \leq k \rangle$ be its least proper initial segment. By the disjointness property (1) with $s_{0, k+1} = s_{0, k}$,

\[|R_1[X]| \geq |u_0[X \cap s_{0, k}^B] \cup R_1[X \setminus s_{0, k}^B]| \]
\[= |u_0[X \cap s_{0, k}^B]| + |R_1[X \setminus s_{0, k}^B]| \]
\[\geq |X \cap s_{0, k}^B| + |X \setminus s_{0, k}^B| = |X|,\]

where the first inequality holds since $u_0 - \{b_0, u_0(b_0)\} \subseteq R_1$ and the last inequality holds since $u_0$ is single-valued, $R$ satisfies the Hall condition, and $R_1[X \setminus s_{0, k}^B] = R[X \setminus s_{0, k}^B] \setminus u_0[s_{0, k}] = R[X \setminus s_{0, k}^B]$ follows from the disjointness property (1).

Otherwise, i.e., $s_0 = \langle s_{0, j} : j \leq k(1) \rangle$ is proper. If $|X| \geq h(1)$, then $|R_1[X] - |X| \geq |R[X] - |X| | \geq 0$ since the original graph $R(B, G)$ satisfies the expanding Hall condition via $h$. We consider in the case of $|X| < h(1)$. By properness, $a_j := s_{0, j}^B \setminus s_{0, j-1}$ is nonempty for each $j \leq h(1)$. Then, $a_j \cap X = \emptyset$ holds for some $0 < j_1 \leq h(1)$, since $\{s_j : j \leq h(1) \}$ is pairwise disjoint. Fix $j_1$. By the disjointness property (1) with the fact $X \cap s_{0, j} = X \cap s_{0, j-1}$, as in the previous paragraph,

\[|R_1[X]| \geq |u_0[X \cap s_{0, j_1}^B] \cup R_1[X \setminus s_{0, j_1}^B]| \]
\[= |u_0[X \cap s_{0, j_1}^B]| + |R_1[X \setminus s_{0, j_1}^B]| \]
\[\geq |X \cap s_{0, j_1}^B| + |X \setminus s_{0, j_1}^B| = |X|,\]

as desired. \[\square\]
We are now prepared to show by $\Sigma_0^0$ induction that $\forall u \exists v h(u, v)$ holds. For the initial step, it clearly holds by Lemma 2.10. We consider the induction step. Let $\theta(\bar{u}, v)$ hold. By Claim 2.11, the remaining graph $R - \{(b_j, (\bar{u}), (b_j)) : j < v + 1\}$ satisfies the Hall condition. Define $h : \mathbb{N} \to \mathbb{N}$ as
\[
\begin{cases}
    h(0) = 0, \\
    h(x) = h(x + v + 1) \quad \text{for} \ x > 0.
\end{cases}
\]
By $\Sigma_0^0$ comprehension, such $h$ exists. It is clearly a witness of the computable expanding Hall condition for our remaining graph. By Lemma 2.10, there exists an appropriate chainable matching $\tilde{m}$, and it is easy to see that $\theta(\tilde{u}, \tilde{m}, v + 1)$ holds. This completes the proof of our theorem.

As an immediate consequence from Theorem 2.7, we can see that $B'_M(G''\cdot M)$ is also provable in $\text{RCA}_0 + \Sigma_0^0$-IND, whereas we do not know whether $\Sigma_0^0$-IND is necessary to prove $B'_M(G''\cdot M)$ and $B'_M(G''\cdot M)$. We also note that the construction of a solution in the proof of Theorem 2.7 is uniform. Hence, the sequential version of $B'_M(G''\cdot M)$ is provable in $\text{RCA}_0 + \Sigma_0^0$ by imitating the proof of Theorem 2.7.

### 2.3. Summary of Section 2.

**Remark 2.12.** As a conclusion from Lemma 2.1 and Lemma 2.4, and Theorem 2.7, it turns out that the computable expanding Hall condition and being $G$-highly recursive are essential for a computable marriage problem to have a computable solution.

**Theorem 2.13.** All of the following assertions are equivalent to $\text{WKL}$ over $\text{RCA}_0$.

\[
B'_M(G\cdot M) \quad B'_M(G\cdot M) \quad B'_M(G\cdot M) \\
B'_M(G'\cdot M) \quad B'_M(G'\cdot M) \quad B'_M(G'\cdot M) \\
B'_M(G''\cdot M) \quad B'_M(G''\cdot M) \quad B'_M(G''\cdot M)
\]

**Proof.** It is clear by the fact that $\text{WKL}_2 \vdash B'_M(G\cdot M)$ [13, Theorem 2.3], Lemma 2.1 and Lemma 2.4 since the computable expanding Hall condition and being highly recursive are restrictions of the expanding Hall condition and being locally finite respectively.

**Proposition 2.14.** $\text{ACA}_0 \vdash B'_M(G\cdot M)$, that is, the following is provable within $\text{ACA}_0$: If $R(B, G)$ is a bipartite graph which satisfies the expanding Hall condition, then $R(B, G)$ has a solution.

**Proof.** Straightforward by a routine inspection of [17, Theorem 6].

**Theorem 2.15.** All of the following assertions are equivalent to $\text{ACA}_0$ over $\text{RCA}_0$.

\[
B'_M(G\cdot M) \quad B'_M(G\cdot M) \quad B'_M(G\cdot M) \\
B'_M(G'\cdot M) \quad B'_M(G'\cdot M) \quad B'_M(G'\cdot M) \\
B'_M(G''\cdot M) \quad B'_M(G''\cdot M) \quad B'_M(G''\cdot M)
\]

**Proof.** It is clear by the fact that $\text{ACA}_0 \vdash B'_M(G\cdot M)$ [13, Theorem 2.2], Proposition 2.14, Lemma 2.3 and Lemma 2.6.

### 3. Marriage Theorems with Constant Bounded Hall Condition

#### 3.1. Hall Condition with Constant Bound.

In this section, we study reverse mathematics of marriage theorems with the constant bounded Hall condition $H_{cb}$ which is introduced in Subsection 1.3. The constant bounded Hall condition means that there are very few choices of partners of each boy. Of course, it guarantees $B$-local finiteness $B'$. However, it does not help to make a computable marriage problem $B$-highly recursive. Nevertheless, the next theorem states that the constant bounded Hall condition renders marriage problems solvable in $\text{RCA}_0$.

**Theorem 3.1.** $\text{RCA}_0 \vdash B_{H_{cb}} G\cdot M$, that is, the following is provable within $\text{RCA}_0$: If $R(B, G)$ is a bipartite graph which satisfies the constant bounded Hall condition, then $R(B, G)$ has a solution.

**Proof.** We reason in $\text{RCA}_0$. Let $\Phi(c) \equiv \forall X \subseteq n B(\langle R[X]\rangle \le \langle X\rangle + c)$.

Note that the statement $\langle R[X]\rangle \le \langle X\rangle + c$ is written as a $\Pi^0_1$ formula, and then so is $\Phi(c)$. Since $R(B, G)$ satisfies the constant bounded Hall condition, $\exists c \Phi(c)$ holds. By $\Pi^0_1$ least number principle, which can be carried out in $\text{RCA}_0$, there exists a least $c_1$ such that $\Phi(c_1)$ holds. It follows from the leastness that there exists $X_i \subseteq n B$ such that $\langle R[X_i]\rangle = \langle X_i\rangle + c_1$. We fix such $X_i$. Then the set $R[X_i]$ exists by $\Sigma_0^0$ comprehension, and $\langle R[X_i]\rangle < \infty$. We first note the following:

\[
(2) \quad \forall b \in B \setminus X_i, \text{there is at most one } g \in R[b] \setminus R[X_i].
\]
since if not, \(|R[X_1 \cup \{b\}]| \geq |X_1 \cup \{b\}| + c_1 + 1\). Moreover, we claim that
\[X_2 := \{b \in B \setminus X_1 : R[b] \subseteq R[X_1]\}\]
is finite, hence exists by bounded \(\Pi^0_1\) comprehension in \(\text{RCA}_0\) (cf. [19, Theorem II.3.9]). Indeed, \(X_2\) has at most \(c_1\) many elements. Otherwise, for such a finite set \(X'\) of size \(c_1 + 1\) with \(R[X'] \subseteq R[X_1]\), we have \(|X_1 \cup X'| \geq |X_1| + c_1 + 1 > |R[X_1]| = |R[X_1] \cup X'|\). Next, we claim that
\[Y_1 := \{g \in G \setminus R[X_1] : |R^{-1}[\{g\}] \setminus X_1| \geq 2\}\]
is finite (actually, of size at most \(c_1\)), and exists by bounded \(\Sigma^0_1\) comprehension in \(\text{RCA}_0\) [19, Theorem II.3.9]. Suppose not. Then there exists a finite set \(Y'\) of such girls that \(|Y'| = c_1 + 1\). Moreover, by (2) with \(Y' \cap R[X_1] = \emptyset\), for every different \(g_1, g_2 \in Y'\), two sets \(R^{-1}[g_1] \setminus X_1\) and \(R^{-1}[g_2] \setminus X_2\) are disjoint. Then \(R^{-1}[Y'] \geq 2|Y'| = 2(c_1 + 1)\) follows. Let \(X'\) be a finite subset of \(R^{-1}[Y']\) such that \(|X'| \geq 2(c_1 + 1)\). By (2), each boy in \(X'\) knows just one girl in \(Y'\).

Therefore,
\[|R[X_1 \cup X']| \leq |R[X_1]| + |Y'| = (|X_1| + c_1) + (c_1 + 1) = |X_1| + 2c_1 + 1.\]

On the other hand,
\[|X_1 \cup X'| = |X_1| + |X'| \geq |X_1| + 2c_1 + 2.\]

These contradict the Hall condition.

Now note that the condition (2) implies \(R[R^{-1}[Y_1]] \subseteq R[X_1] \cup Y_1\). Therefore, we have
\[|X_1| + |R^{-1}[Y_1]| = |X_1 \cup R^{-1}[Y_1]| \leq |R[X_1] \cup R^{-1}[Y_1]| \leq |R[X_1] \cup Y_1| = |R[X_1]| + |Y_1| = (|X_1| + c_1) + c_1 = |X_1| + 2c_1.\]

Hence, \(R^{-1}[Y_1]\) has at most \(2c_1\) many elements, and \(R^{-1}[Y_1]\) exists by \(\Sigma^0_1\) comprehension. Moreover, \(|X_1 \cup X_2 \cup R^{-1}[Y_1]|\) is finite. On the other hand, \(R[X_1 \cup X_2 \cup R^{-1}[Y_1]] \subseteq R[X_1] \cup Y_1\) holds by our choice of \(X_1, X_2\) and \(Y_1\). Thus the following finite subgraph
\[(X_1 \cup X_2 \cup R^{-1}[Y_1], R[X_1] \cup Y_1; R)\]
of \((B, G)\) satisfies the Hall condition because of the Hall condition for the original graph \((B, G)\). Then it has a matching \(M\) by the finite marriage theorem in \(\text{RCA}_0\) ([13, Theorem 2.1]). Again by (2), each boy \(b \notin X_1 \cup X_2 \cup R^{-1}[Y_1]\) knows just one girl \(g_b \notin R[X_1] \cup Y_1\). Moreover, for any such boys \(b\) and \(b'\), if \(b \neq b'\), then \(g_b \neq g_{b'}\), since \(b, b' \notin R^{-1}[Y_1]\). Therefore, \(M \cup \{(b, g_b) : b \in B \setminus (X_1 \cup X_2 \cup R^{-1}[Y_1])\}\) is a solution of \((B, G)\) in \(\text{RCA}_0\). This completes the proof of our theorem. \(\square\)

Consequently, all of \(B_{H_n}^r G^r-M, B_{H_n}^r G^r-M, B_{H_n}^r G^r-M\) and \(B_{H_n}^r G^r-M\) are probable in \(\text{RCA}_0\). As a corollary, if a computable bipartite graph \((B, G)\) satisfies the constant bounded Hall condition, then \((B, G)\) has a computable solution. However, note that the algorithm in the proof of Theorem 3.1 to give a solution for a given instance of \(B_{H_n}^r G^r-M\) is not uniform, in contrast to the uniformity of the algorithm in the proof of Theorem 2.7 for \(B_{H_n}^r G^r-M\).

3.2. Sequential Reverse Mathematics.

3.2.1. Extracting Non-uniformity from Proofs. Our proof of Theorem 3.1 in \(\text{RCA}_0\) contains an implicit non-uniformity in the use of least number principle. The next theorem suggests that this non-uniformity can not be avoided. We use a notation \(\text{Seq}(A)\) for the sequential version of \(A\) below.

**Theorem 3.2.** The following are pairwise equivalent over \(\text{RCA}_0\).

1. ACA.
2. \(\text{Seq}(B_{H_n}^r G^r-M)\), that is, for all sequence \(\langle B_n, G_n, R_n, k_n \rangle_{n \in \mathbb{N}}\) such that \(R_n(B_n, G_n)\) satisfies the constant bounded Hall condition via \(k_n\), there exists a sequence \(\langle M \rangle_{n \in \mathbb{N}}\) of the solutions.\(^1\)
3. \(\text{Seq}(B_{H_n}^r G^r-M)\), that is, for all sequence \(\langle B_n, G_n, R_n, k_n \rangle_{n \in \mathbb{N}}\) such that \(R_n(B_n, G_n)\) is \(G_n\)-locally finite and satisfies the constant bounded Hall condition via \(k_n\), there exists a sequence \(\langle M \rangle_{n \in \mathbb{N}}\) of the solutions.\(^2\)

Before proving this theorem, we will show the following related equivalences with \(\text{WKL}\).

**Theorem 3.3.** The following are pairwise equivalent over \(\text{RCA}_0\).

1. WKL.
2. \(\text{Seq}(B_{H_n}^r G^r-M)\), that is, for all sequence \(\langle B_n, G_n, R_n, p_n, k_n \rangle_{n \in \mathbb{N}}\) such that \(R_n(B_n, G_n)\) is \(B_n\)-highly recursive via \(p_n\) and satisfies the constant bounded Hall condition via \(k_n\), there exists a sequence \(\langle M \rangle_{n \in \mathbb{N}}\) of the solutions.\(^3\)

---

\(^1\)Note that the sequence \(k_n\) is included in the sequence of marriage problems. This is appropriate sequentialization because our focus is on the non-uniformity of the construction of a solution from the given constant bounded Hall condition via \(k\). (cf. [8])

\(^2\)Note that the sequence \(p_n\) is included in the sequence of marriage problems. This is the appropriate sequentialization for our purpose as in the previous footnote. (cf. [8])
(3) \( \text{Seq}(B'^*_R, G^*\cdot M) \), that is, for all sequence \( (B_n, G_n, R_n, p_n, k_n)_{n \in \mathbb{N}} \) such that \( R_n(B_n, G_n) \) is \( B_n \)-highly recursive via \( p_n \), \( G_n \)-locally finite and satisfies the constant bounded Hall condition via \( k_n \), there exists a sequence \( (M_i)_{i \in \mathbb{N}} \) of the solutions.

Proof. \((1 \rightarrow 2) \) holds by the facts that \( \text{WK}_0 \vdash B'^*_R \cdot G^* \cdot M \) ([13, Theorem 2.3]) and \( \text{RCA}_0 \vdash \text{WK}_1 \leftrightarrow \text{Seq}(\text{WK}_1) \) ([14, Lemma 5]). \((2 \rightarrow 3) \) is trivial. We shall show \((3 \rightarrow 1) \). It suffices to separate the ranges of disjoint functions ([19, Lemma IV.4.4]). Let \( f, g : \mathbb{N} \rightarrow \mathbb{N} \) be given injections with disjoint ranges.

We construct a sequence of bipartite graphs \( (R_n(B_n, G_n))_{n \in \mathbb{N}} \) in \( \text{RCA}_0 \). For each \( n \in \mathbb{N} \), put \( B_n = G_n = \mathbb{N} \). At first, \( (0,0) \) and \( (0,1) \) are enumerated into each \( R_n \). At the \( j \)-th step in the construction of \( R_n \), if \( f(j) = j \) occurs, then put \( (j+1,1) \in R_n \). If \( g(j) = j \) occurs, then put \( (j+1,0) \in R_n \). Otherwise, put \( (j+1,j+2) \in R_n \).

We put \( (p_n)_{n \in \mathbb{N}} := (p)_{n \in \mathbb{N}} \) where \( p : \mathbb{N} \rightarrow \mathbb{N} \) such that \( p(n) = n+1 \) and \( (k_n)_{n \in \mathbb{N}} := (1)_{n \in \mathbb{N}} \) in \( \text{RCA}_0 \). Then each \( n \) graph \( R_n(B_n, G_n) \) is \( G_n \)-locally finite and \( B_n \)-highly recursive via \( p_n \), and it is also easy to see that for all \( n \) and \( X \subseteq B_n \), \( |X| \leq |R_n[X]| \leq |X| + k_n \) holds within \( \text{RCA}_0 \). Then \( \text{Seq}(B'^*_R, G^* \cdot M) \) implies the existence of a sequence \( (M_i)_{i \in \mathbb{N}} \) of solutions for \( (R_n(B_n, G_n))_{n \in \mathbb{N}} \). Define \( V := \{ i : (0,0) \in M_i \} \) by \( \Sigma^0_0 \) comprehension. Then \( V \) separates the ranges of \( f \) and \( g \) because of the above construction. \( \square \)

Proof of Theorem 3.2. \((1 \rightarrow 2) \) is shown straightforwardly by revising the proof of \( \text{ACA}_0 \vdash B'^*_R \cdot G^* \cdot M \) by Hirst ([13, Theorem 2.2]) a bit. \((2 \rightarrow 3) \) is trivial. We show \((3 \rightarrow 1) \) by revising a proof of \((3 \rightarrow 1) \) of Theorem 3.3 by using “liberation method” as the proofs of Lemma 2.3 and Lemma 2.6.

Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be an injection and for each \( n \in \mathbb{N} \), put \( B_n = G_n = \mathbb{N} \). First, put \( (0,0), (0,1), (1,0) \in R_n \). At the \( j \)-th step in the construction of \( R_n \), if \( f(j) = i \) occurs, then put \( (j+2,1), (j+2,j+2) \in R_n \). Otherwise, put \( (j+2,j+2) \in R_n \).

We put \( (p_n)_{n \in \mathbb{N}} := (p)_{n \in \mathbb{N}} \) where \( p : \mathbb{N} \rightarrow \mathbb{N} \) such that \( p(n) = n+1 \) and \( (k_n)_{n \in \mathbb{N}} := (1)_{n \in \mathbb{N}} \) in \( \text{RCA}_0 \). Then each \( n \) graph \( R_n(B_n, G_n) \) satisfies our assumptions, so has a sequence \( (M_i)_{i \in \mathbb{N}} \) of solutions by \( \text{Seq}(B'^*_R, G^* \cdot M) \). It is easy to see that \( V := \{ i : (0,0) \in M_i \} \) is the range of \( f \).

Remark 3.4. It has been recently established in [15] and [5] that for a \( \Pi^1_2 \) statement of a certain syntactical form, its provability in (semi-)intuitionistic systems guarantees the provability of its sequential form in weak subsystems of second order arithmetic.\(^3\) Such kind of results are called “unification theorems”. The first unification theorems in higher order setting [15] can be applied for \( \Pi^1_2 \) statements of the following syntactical form:

\[
(\alpha) \quad \forall X \left( \varphi(X) \rightarrow \exists Y \psi(X,Y) \right),
\]

where \( \varphi(X) \) is purely universal and \( \psi(X,Y) \) is in sufficiently large class \( \Gamma_2 \) of formulas. On the other hand, Dorais has recently shown other unification theorems in second order setting [5]. The advantage of Dorais’s unification theorems compared to the former is that it can be applied for more \( \Pi^1_2 \) statements, namely, for \( \Pi^1_2 \) statements of the form \( (\alpha) \) with \( \varphi(X) \) including purely existential formulas as subformulas. (See [5, Section 4] for details.) By a careful inspection, one can check that the assertion “a bipartite graph \( (B, G; R) \) satisfies the constant bounded Hall condition via \( k^* \)” is (in intuitionistic sense) formalized as a formula of form \( \forall X \exists Y A_0(B, G, R, k) \) where \( A_0 \) is quantifier-free. That is to say, Dorais’s unification theorems can be applied to our marriage theorems with the constant bounded Hall condition while the unification theorems in [15] can not. As a consequence of Theorem 3.2 and 3.3, we have the following. (Note that \( \text{EL} \) is the intuitionistic second order system and \( \text{RCA}_0 \) consists of \( \text{EL} \) and the law of excluded middle. See [5] for the definition of each symbol.)

\begin{align*}
(1) & \quad B'^*_R \cdot G^* \cdot M \text{ and } B'^*_L \cdot G^* \cdot M \text{ are not provable in } \text{EL} + \text{WKL} + \text{GC}_L + \text{CN}_L. \\
(2) & \quad B'^*_R \cdot G^* \cdot M \text{ and } B'^*_L \cdot G^* \cdot M \text{ are not provable in } \text{EL} + \text{GC} + \text{CN}. 
\end{align*}

3.2.2. Recursive Construction. By inspecting the proofs of Theorem 3.2 and Theorem 3.3, one notices that even if the Hall condition is bounded by \( k = 1 \), the marriage problem is not solvable uniformly in \( \text{RCA}_0 \). By contrast, if the marriage problem is \( G \)-highly recursive, then it is uniformly solvable in \( \text{RCA}_0 \), regardless of the size of the constant bound on the Hall condition.

Theorem 3.5. \( \text{RCA}_0 \vdash \text{Seq}(B'^*_R, G^* \cdot M) \), that is, the following is provable in \( \text{RCA}_0 \): For any sequence \( (B_n, G_n, R_n, p_n, k_n)_{n \in \mathbb{N}} \) such that \( R_n(B_n, G_n) \) is \( G_n \)-highly recursive via \( p_n \) and satisfies the constant bounded Hall condition via \( k_n \), there exists a sequence \( (M_i)_{i \in \mathbb{N}} \) of the solutions.

Proof. Rather than formally proving the sequential form, we give a uniform proof in \( \text{RCA}_0 \) of \( B'^*_R \cdot G^* \cdot M \) for a graph with \( B \) infinite. This proof can easily be transformed into a proof of \( \text{Seq}(B'^*_R, G^* \cdot M) \) in \( \text{RCA}_0 \).

Let \( R(B, G) \) be a bipartite graph which is \( G \)-highly recursive and satisfies the constant bounded Hall condition via \( k \), and \( \{ b_i : i \in \mathbb{N} \} \) be an enumeration of \( B \). We shall now construct a solution of \( R(B, G) \) by a recursive procedure. Let \( \theta(u, v) \) say that \( u \) encodes a sequence \( \{ u_i \}_{i \leq v+1} \) of length \( v+1 \) of chains \( u_i = (u^i_{b_1}, u^i_{b_2})_{b_i \leq i \leq \theta(u_i)} \), where each \( u_i \) is a least non-proper \( R \)-chain (Definition 2.9) of finite length in the remaining graph \( R(B, G) = R(B, G) - \bigcup_{i < v} u_i \).
and $b_i$ is contained in $\bigcup_{j \leq i} u_i^B$. Note that the present $u_i$ is not a chainable matching as in the proof of Theorem 2.7. Now $\theta(u,v)$ is written as a $\Sigma_0^0$ formula because $R(B,G)$ is $G$-highly recursive.

Suppose that we have shown $\forall v \exists u \theta(u,v)$. Then there exists a function which outputs the unique $u^v$ for each $v \in N$ by $\Delta_1^0$ comprehension as in the proof of Theorem 2.7. Now we construct a function $g$ by the following primitive recursion in RCA$_0$.

$$g(0) = \text{the least matching of } (u^0)_0^B \text{ in } R,$$

$$g(v + 1) = \text{the least matching of } (u^{v+1})_{v+1}^B \text{ in } R - \bigcup_{u \leq v} u_i.$$

This primitive recursion works by the finite marriage theorem, since the definition of $\theta(u,v)$ ensures the Hall condition for each subgraph $(u^{v+1})_{v+1}$ in each remaining graph. We take $M := \bigcup_{v \leq u} g(v)$, then we can straightforwardly verify in RCA$_0$ that $M$ is an injection from $B$ to $G$. Thus, it suffices to show $\forall v \exists u \theta(u,v)$ by $\Sigma_0^0$ induction on $v$. To show $\exists u \theta(u,0)$, we first show the following key claim.

**Claim 3.6 (RCA$_0$).** If a bipartite graph $R(B,G)$ satisfies the constant bounded Hall condition via $k \in N$, then there is no proper $R$-chain $s = (s_i^B, s_i^G)_{i < \theta(n)}$ of length more than $t(k+1)$, where $t(k) := (k(k+3))/2$.

*(Proof of Claim.)* Suppose not, i.e., assume that $s = (s_i^B, s_i^G)_{i < \theta(n)}$ be a proper $R$-chain of length more than $t(k+1).$ Note that $t(k+1) - (t(k) + 1) = k$. Now we shall show that for all $n \leq k+1$ there exists $X \subseteq s_{t(n)}^B$ and $Y \subseteq s_{t(n)}^G$ such that $Y \subseteq R[X]$ and $|X| + n \leq |Y|$ by induction on $n$. Note that the above statement can be written as a $\Sigma_0^0$ formula with the use of $s$, then this induction can be carried out in our system RCA$_0$. The initial step is accomplished obviously. Let $X_n$ and $Y_n$ be witnesses of the case of $n$, i.e., $X_n \subseteq s_{t(n)}^B, Y_n \subseteq s_{t(n)}^G, Y_n \subseteq R[X_n]$ and $|X_n| + n \leq |Y_n|$ hold. By the properness of $R$-chain $s$ (see Fig. 4), we can choose $g_j \in s_{j+1}^G \setminus s_{j-1}^G \neq \emptyset$ for each $t(n) < j \leq t(n+1)$.

- In the case that $s_{t(n) + 1}^B \cap R^{-1}[g_j] = \emptyset$ for some $t(n) + 1 < j \leq t(n+1)$.

  Hence, $R[s_{t(n) + 1}^B \cap \{g_j\}] = \emptyset$. Then, $g_i \in s_{t(n) + 1}^G \subset R[s_{t(n) + 1}^B]$ implies that there is $b \in s_{j-1}^B \setminus s_{t(n) + 1}^B$ such that $g_{j-1} \in R[b]$. Now $b \in s_{j-1}^B \setminus s_{t(n) + 1}^B = R^{-1}[s_{t(n) + 1}^B \setminus R^{-1}[s_{t(n)}^G]]$ implies that there is $\hat{g} \in s_{j-1}^G \setminus s_{t(n)}^G$ such that $\hat{b} \in R^{-1}(\hat{g})$. As $g_i \notin s_{j-1}^G$, the girls $g_i$ and $\hat{g}$ are different, and they are not contained in $s_{t(n)}^G$. Hence, the boy $\hat{b} \notin X_n \subseteq s_{t(n)}^B$ knows two different girls $g_{j-1}, \hat{g} \notin Y_n \subseteq s_{t(n)}^G$. Therefore, for $X_{n+1} := X_n \cup \{b\} \text{ and } Y_{n+1} := Y_n \cup \{g_{j-1}, \hat{g}\}$, we have $Y_{n+1} \subseteq R[X_{n+1}]$ and $|X_{n+1}| + n + 1 \leq |Y_{n+1}|$.

- Otherwise, $s_{t(n) + 1}^B \cap R^{-1}[g_j] \neq \emptyset$ for every $t(n) + 1 < j \leq t(n+1)$.

Choose $x_j \in s_{t(n) + 1}^B \cap R^{-1}[g_j]$ for each $t(n) + 1 < j \leq t(n+1)$, and put $X = \{x_j\}_{t(n) + 1 < j \leq t(n+1)}$. Since $(s_{t(n) + 1}^B, s_{t(n) + 1}^G)$ satisfies the Hall condition, there exists $\hat{Y} \subseteq s_{t(n) + 1}^G$ such that $|X| \leq |\hat{Y}|$ holds. Then one can verify that $X$ and $\hat{Y} \cup \{g_j\}_{t(n) + 1 < j \leq t(n+1)}$ are witnesses of the case of $n + 1$ straightforwardly.

Therefore the induction step is also accomplished. Then there exists $X \subseteq s_{t(k)}^B$ and $Y \subseteq s_{t(k)}^G$ such that $Y \subseteq R[X]$ and $|X| + k < |Y|$ holds. This contradicts our assumption that $R(B,G)$ satisfies the constant bounded Hall condition via $k$ and completes the proof of our claim. \[\square\]
Because \( R(B,G) \) is \( G \)-highly recursive, we can effectively produce a non-proper \( R \)-chain \( s \) with starting point \( b_0 \in B \) by the following procedure: Let \( s_0^B \) be the set consisting only of \( b_0 \), take the first witnessed set of girls \( s_1^G \) such that \( \langle s_j^B, s_j^G \rangle_j \leq 3 \) forms an \( R \)-chain, and put \( s_{j+1}^B = R^{-1}[s_j^G] \). Claim 3.6 ensures that this procedure would stop eventually until \( j \) is up to \( t(k+1) \), i.e., \( \langle s_j^B, s_j^G \rangle_j \leq t(k+1) \) is non-proper. Then, by \( \Sigma^0_0 \) least number principle, there exists \( u_0 \) such that \( \theta(u_0,0) \) holds. Thus the initial step is accomplished.

Next we turn to the induction step. Assume that \( \exists u \theta(u,v) \) holds, and let \( u' \) be \( u \) such that \( \theta(u,v) \) holds. Then \( R' = R - \bigcup_{u \subseteq u'} s_j \) satisfies the constant bounded Hall condition by the disjoint property (1). As in the initial step, we can effectively produce a non-proper \( R' \)-chain \( s' \) of finite length, where we take \( b_{e+1} \) as the starting point of \( s' \) if \( b_{e+1} \notin (u')^B \). Let \( u_{e+1} \) be such a least \( s' \), then \( \theta \left( u', u_{e+1}, v + 1 \right) \) holds. This completes the proof of our theorem. \( \square \)

**Corollary 3.7.** \( \text{Seq}(B_{H,H}, G'' \cdot M) \) is provable in \( \text{RCA}_0 \).

### 4. Weihrauch Degrees

#### 4.1. Basic Terminology.**

Many theorems of mathematics can be formalized as \( \Pi^1_1 \) sentences. In particular, marriage theorems can be written as the following \( \Pi^1_1 \) sentences:

\[
(\forall R) \left( \varphi(R) \rightarrow \exists M \psi(R, M) \right),
\]

where \( \varphi(R) \) denotes that \( R \) is a graph with \( B^{(1)}, G^{(1)}, H^{(1)} \), and \( \psi(R, M) \) denotes that \( M \) is a matching of \( R \). Importantly, such a \( \Pi^1_1 \) theorem can be viewed as a (partial) multi-valued function \( f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \) with an arithmetical domain, by interpreting \( \psi(R, M) \) as \( M \in f(R) \). Every single-valued selection \( s \) of \( f \) can be thought of as a witness of the \( \Pi^1_1 \) theorem, that is, \( s \) witnesses \( \psi(R, s(R)) \). The following reducibility notion is useful to estimate how difficult it is to find a witness of a given \( \Pi^1_1 \) theorem.

**Definition 4.1.** (See also [2, 3].) For multi-valued functions \( f \) and \( g \), \( f \) is \textit{Weihrauch reducible to} \( g \) (denoted \( f \leq_W g \)) if there are computable functions \( H, K \) such that \( K(\langle Id, GH \rangle) \) is a single-valued selection of \( f \) for every single-valued selection \( G \) of \( g \).

Let us consider the following partial multi-valued functions.

\[
\begin{align*}
B_{H,H}^{(1)} G^{(1)} M(R) &= \{ M : M \text{ is a matching of } R \}, \\
\text{dom}(B_{H,H}^{(1)} G^{(1)} M) &= \{ R : R \text{ is a graph with } B^{(1)}, G^{(1)}, H^{(1)} \}, \\
\text{KL}(T) &= \text{WKL}(T) = \{ P : P \text{ is an infinite path through } T \}, \\
\text{dom}(\text{KL}) &= \{ T \subseteq \mathbb{N}^* : T \text{ is an infinite finitely branching tree} \}, \\
\text{dom}(\text{WKL}) &= \{ T \subseteq \mathbb{N}^* : T \text{ is an infinite binary tree} \}, \\
\text{Lim}_X(p_n) &= \lim_{n \to \infty} p_n, \text{ where } X \text{ is a topological space}, \\
\text{dom}(\text{Lim}_X) &= \{ (p_n)_{n \in \mathbb{N}} : X^\mathbb{N} : \lim p_n \text{ converges} \}.
\end{align*}
\]

Intuitively, \( f \leq_W g \) means that, to find a solution to the problem \( f(x) \), it suffices to find a solution \( y \) to \( g(H(x)) \), since \( K(x, y) \) is a solution to \( f(x) \). The Weihrauch degree of the identity function \( \text{Id} \) is analogous to \( \Delta^0_1 \) comprehension axiom \( \text{RCA} \) in second order arithmetic. The limit function \( \text{Lim} \) is analogous to the \( \Sigma^0_1 \) comprehension, that is equivalent to the arithmetical comprehension \( \text{ACA} \) in reverse mathematics. Every function \( f \leq_W \text{Lim} \) is said to be \textit{computable with finitely many mind changes}, and \( f \leq_W \text{Lim} \) is also said to be \textit{limit computable} [1]. The function \( \text{Lim} \) is also known as the \textit{discrete limit} [1]. The \textit{parallelization} of a partial multi-valued function \( f \) is defined by \( f((x_i)_{i \in \mathbb{N}}) = \prod_{i \in \mathbb{N}} f(x_i) \). Given a multi-valued function \( f \) with an associated \( \Pi^1_1 \) theorem \( \tau \), its parallelization \( f^\tau \) can be seen as the sequential version \( \text{Seq}(\tau) \) of the original theorem \( \tau \). As shown in [2], \( \text{WKL} \) is Weihrauch equivalent to \( \text{LLPO} \). As a counterpart of this result in the context of \textit{constructive reverse mathematics} over intuitionistic analysis \( \text{El} \) or intuitionistic finite type arithmetic \( \text{HA}^\omega \), Ishihara [16] has shown that \( \text{WKL} \) is equivalent to \( \text{LLPO} \) plus the axiom of countable choice for \( \Pi^1_1 \) disjunctions, \( \Pi^1_1\text{-AC}^\omega \). It is also known that \( \text{Lim} \equiv_W \text{LLPO} \equiv_W \text{Lim} \) holds (see [2]).

#### 4.2. Weihrauch Degrees inside ACA.

Contrary to the inequality \( \text{Lim} \leq_W \text{KL} \) (obtained from the fact that \( \text{KL} \) is equivalent to weak König’s lemma relative to the jump), the standard reverse mathematics [19] does not distinguish \( \text{Lim} \) from \( \text{KL} \) since the collection of functions \( \text{ACA} \) is closed under composition. Therefore, even if some theorem is shown to be equivalent to \( \text{ACA} \) over \( \text{RCA}_0 \), the exact computational strength of the theorem has many possibilities including the jump, the double jump, and so on. The concept of Weihrauch degrees may help us to better understand the computational strength of \( \Pi^1_1 \) theorems.
**Theorem 4.2.**  
(1) All of the following multi-valued functions are Weihrauch equivalent to KL.
\[ \mathbb{B}_{Ht}^G \cdot M \quad \mathbb{B}_{Ht}'^G \cdot M \quad \mathbb{B}_{Ht}'^G' \cdot M \]

(2) All of the following multi-valued functions are Weihrauch equivalent to \( \text{Lim}_{\geq} \).
\[ \mathbb{B}_{Ht}^G \cdot M \quad \mathbb{B}_{Ht}'^G \cdot M \quad \mathbb{B}_{Ht}'^G' \cdot M \]

(3) All of the following multi-valued functions are Weihrauch equivalent to WKL.
\[ \mathbb{B}_{Ht}^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}'^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}'^{G^\prime} \cdot M \]

(4) All of the following multi-valued functions are Weihrauch equivalent to id.
\[ \mathbb{B}_{Ht}^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}'^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}'^{G^\prime} \cdot M \]

**Proof.** (1) It suffices to show that \( \mathbb{B}_{Ht}^G \cdot M \leq_W \text{KL} \leq_W \mathbb{B}_{Ht}^{G^\prime} \cdot M \). It is easy to see that \( \mathbb{B}_{Ht}^G \cdot M \leq_W \text{KL} \) since the set of all injective selections for a locally finite multi-valued function \( F \) forms a bounded \( \Pi^0_1 \) \( \mathbb{F} \) class. Conversely, given finitely branching tree \( T \), we can effectively construct an instance of \( \mathbb{B}_{Ht}^{G^\prime} \cdot M \) whose solutions are computably homeomorphic to \( \text{KL}(T) \) as follows. Put \( B = T, G = T \setminus \{ \emptyset \} \), and for each \( \sigma \in T \setminus \{ \emptyset \} \) enumerate \( (\sigma, \sigma) \in R \) and \( (\sigma^-, \sigma) \in R \), where \( \sigma^- \) is the unique immediate predecessor of \( \sigma \).

(3) Straightforward. (4) By uniformity of the proof of the Theorem 2.7.

(2) By using \( \text{Lim}_{\geq} \), \( \mathbb{B}_{Ht}^G \cdot M \) is reducible to \( \mathbb{B}_{Ht}^{G^\prime} \cdot M \). Thus, by (4), we have \( \mathbb{B}_{Ht}^G \cdot M \leq_W \text{Lim}_{\geq} \). Conversely, by uniformity of proofs of Lemma 2.3 and 2.6, \( \text{Lim}_{\geq} \) is Weihrauch reducible to \( \mathbb{B}_{Ht}^{G^\prime} \cdot M \) and \( \mathbb{B}_{Ht}'^{G^\prime} \cdot M \).

**4.3. Weihrauch Degrees inside RCA.** If a \( \Pi^1_3 \) theorem \( \tau \) is provable in RCA, then the associated multi-valued function \( f_\tau \) is always expected to be non-uniformly computable. Here, a multi-valued function \( f : \mathbb{N}^n \rightharpoonup \mathbb{N}^m \) is said to be non-uniformly computable if there is a single-valued selection \( F \) of \( f \) such that \( F(x) \) is computable in \( x \) for all \( x \in \text{dom}(F) \). For instance, \( \text{Lim}_{\geq} \), LPO, LLPO, and id are non-uniformly computable. Over RCA, if its sequential version \( \text{Seq}(\tau) \) is equivalent to an axiom, say WKL, then one may also guess that its parallelization \( \tilde{f}_\tau \) is expected to be Weihrauch-equivalent to WKL. For instance, LLPO is a such a multi-valued function, that is, LLPO is non-uniformly computable, and its parallelization \( \text{LLPO} \) is Weihrauch equivalent to WKL. Unfortunately, however, LLPO is not a unique such function.

For a tree \( T \), a string \( \sigma \in T \) is branching if at least two immediate successors of \( \sigma \) are contained in \( T \). Now we introduce the following multivalued functions.
\[ \text{WKL}_{\leq}(T) = \text{WKL}_{\geq}(T) = \text{WKL}(T), \]
\[ \text{dom}(\text{WKL}_{\geq}(n)) = \{ T \in \text{dom}(\text{WKL}) : T \text{ has less than } n \text{ branching nodes} \}, \]
\[ \text{dom}(\text{WKL}_{\leq}(n)) = \{ T \in \text{dom}(\text{WKL}) : T \text{ has only } n \text{ branching nodes} \}. \]

Obviously LLPO \( \leq_W \text{WKL}_{\geq} \), and every WKL\( _{\geq} \) is Weihrauch reducible to an iterative use of sufficiently many LLPO's. Hence, one can think of WKL\( _{\leq, \geq} \) as a natural enrichment of LLPO. We now see the Weihrauch degree of constant bounded marriage theorems.

**Theorem 4.3.**  
(1) All of the following multi-valued functions are Weihrauch equivalent to \( \text{Lim}_{\geq} \).
\[ \mathbb{B}_{Ht}^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}'^{G^\prime} \cdot M \]

(2) All of the following multi-valued functions are Weihrauch equivalent to id.
\[ \mathbb{B}_{Ht}^{G^\prime} \cdot M \quad \mathbb{B}_{Ht}'^{G^\prime} \cdot M \]

**Proof.** (1) To see \( \mathbb{B}_{Ht}^G \cdot M \leq_W \text{Lim}_{\geq} \), it suffices to check that the canonical indices of parameters \( c_1, X_1, X_2, Y_1, R[X_1], \) and \( R^{-1}[Y_1] \) in Theorem 3.1 are effectively determined by finitely many mind changes. At stage \( s + 1 \), if we find a finite set \( X \) of boys such that \( |R\!_s[X]| > |X| + c_{1,s}, \) then put \( X_1 = X = X \cup \{ s \} \) and \( X_{2,s} = X \cup \{ s \} \). Clearly, \( c_1 = \lim_{s \rightarrow \infty} c_{1,s} \) and \( X_1 = \lim_{s \rightarrow \infty} X_{1,s} \). Conversely, to see \( \text{Lim}_{\geq} \leq_W \mathbb{B}_{Ht}^{G'} \cdot M \), given a sequence \( p = (p_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N} \), we construct a graph \( R_p \) with the constant bounded Hall condition. Put \( B = G = \mathbb{N} \) and \( (0, 0) \in R \) with a parameter \( g_0 = 0 \). If \( p_{n+1} = p_n \), then put \( (n+1, g_n) \) for all \( n \). Then it is easy to see that \( R \) has the unique matching \( M \), and \( p_{M(0)} = \lim_{s \rightarrow \infty} p_s \).

(2) By the uniformity of the proof of Theorem 3.5.
Theorem 4.4. WKL\(\preceq\omega\) \(B_{\Pi^0_1}^\omega G\)-\(\mathbb{M}\) \(\preceq\omega\) \(B_{\Pi^0_3}^\omega G\)-\(\mathbb{M}\) \(\preceq\omega\) inf\((\text{Lim}_{\mathbb{M}}, \text{WKL})\).

Proof. The inequality \(B_{\Pi^0_1}^\omega G\)-\(\mathbb{M}\) \(\preceq\omega\) inf\((\text{Lim}_{\mathbb{M}}, \text{WKL})\) clearly holds by Theorem 4.3 (1). Given a tree \(T\) and a nonempty string \(\sigma\) \(\in T\), we denote by \(l_T(\sigma)\) the longest initial segment of \(\sigma\) whose immediate predecessor is branching or empty. To see WKL\(\preceq\omega\) \(B_{\Pi^0_1}^\omega G\)-\(\mathbb{M}\), assume that \(T\) is an infinite tree with only finitely many branching nodes. Put \(B = T\) and \(G = T \setminus \{\langle\rangle\}\). For each \(\sigma\) \(\in T\), if \(\sigma\) is not a leaf, then enumerate \((\sigma, \tau) \in R\) for each immediate successor \(\tau\) \(\in T\). If a nonempty string \(\sigma\) \(\in T\) is a leaf or branching, then we also enumerate \((\sigma, l_T(\sigma)) \in R\). Note that every non-branching boy knows just one girl, and every nonempty branching boy knows just three girls. Hence, if \(T\) has only finitely many branching nodes, then \(R(B, G)\) is constant bounded. If \(T\) is of the form \(l_T(\sigma)\) such that \(\sigma\) is a leaf or branching, then \(\tau = l_T(\sigma)\) is known by at most two boys \(\sigma\) and \(l_T(\sigma)^-\). Here note that \(l_T(\sigma)^-\) must be branching. Otherwise, \(\tau\) is known by at most one boy. Hence, \(R(B, G)\) satisfies the Hall condition. Given any matching of \(R(B, G)\), we can effectively find an infinite path through \(T\).

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