THE A_{inf}-COHOMOLOGY IN THE SEMISTABLE CASE

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1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring that has mixed characteristic $(0,p)$ and a perfect residue field k, let A_{inf} be the Fontaine period ring $W(\mathcal{O}^{\flat}_{\widehat{\overline{K}}})$, and let $\mathcal{X}$ be a proper, semistable $\mathcal{O}$-scheme such that $\mathcal{X}_K$ is smooth (the precise setting is given in §1.1). The main goal of this note is to construct an A_{inf}-valued cohomology theory $R\Gamma_{A_{\text{inf}}}(\mathcal{X})$ of $\mathcal{X}$ that interpolates between the integral p-adic étale cohomology and the integral logarithmic de Rham cohomology and to deduce the cohomology specialization results Theorem 4.12 and Corollary 4.13. In the case when $\mathcal{X}$ is smooth, this has been accomplished in [BMS16] by Bhatt–Morrow–Scholze, and we base our construction on their methods. The key differences between our arguments and those of op. cit. are highlighted in the introductions of §§2–3.

We stress that in preparing this note our focus was Theorem 4.12 and that we have neither obtained nor attempted to obtain comprehensive semistable generalizations of the results of [BMS16]—for instance, we assume that $\mathcal{X}$ is defined over $\mathcal{O}$ (instead of $\mathcal{O}_{\widehat{\overline{K}}}$) and that it is a scheme (instead of a formal scheme). More thorough generalizations would, of course, be of interest: e.g., it would be worthwhile to relate $R\Gamma_{A_{\text{inf}}}(\mathcal{X})$ to the Hyodo–Kato cohomology of $\mathcal{X}_k$, extending the relation to the crystalline cohomology in the smooth case.

The approach that we take is to reduce to the results and techniques of [BMS16] whenever possible, so we assume familiarity with certain parts of op. cit. and certain parts of [Sch13] (we give precise references for the results we use). We also assume familiarity with certain auxiliary results from algebraic geometry, rigid geometry, or p-adic Hodge theory, for which we sometimes give references for completeness.

1.1. The setup. The following notational setup is in place throughout this note.

- We fix a mixed characteristic $(0,p)$ complete discretely valued field K whose residue field k is perfect, we let $\mathcal{O} \subset K$ denote the valuation ring.

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- We fix a finite type, separated, flat $\mathcal{O}$-scheme $\mathcal{X}$ such that $\mathcal{X}_K$ is K-smooth and $\mathcal{X}_k$ is a normal crossings divisor in $\mathcal{X}$ (in the sense of [SP, 0BSF]), so that, in particular, $\mathcal{X}$ is regular.

We do not assume at the outset that $\mathcal{X}$ is proper, even though we will impose this later.

Since k is perfect and $\mathcal{X}_k$ is reduced (cf. [SP, 0BIA]), $\mathcal{X}_k$ is generically k-smooth. Moreover, étale locally on $\mathcal{X}$ there exists a regular sequence for which the product of its several first terms cuts out $\mathcal{X}_k$, so the miracle flatness theorem [EGA IV$_2$, 6.1.5] and the perfectness of k ensure that every $x \in \mathcal{X}_k$ has an étale neighborhood $U \to \mathcal{X}$ that admits an étale $\mathcal{O}$-morphism

$$U \to \mathrm{Spec}(\mathcal{O}[t_0, \dots, t_d]/(t_0 \cdots t_r - \pi)) \qquad \text{for some} \quad 0 \leqslant r \leqslant d \quad \text{and a uniformizer} \quad \pi \in \mathcal{O}.$$

For our purposes, it will be more convenient to change coordinates to conclude that every $x \in \mathcal{X}_k$ has an étale neighborhood $U \to \mathcal{X}$ that admits an étale $\mathcal{O}$-morphism

$$U \to \mathrm{Spec}(\mathcal{O}[t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}]/(t_0 \cdots t_r - \pi)) \qquad \text{for some} \quad 0 \leqslant r \leqslant d. \tag{$\square$}$$

- We let $C := \widehat{\overline{K}}$ be the completion of a fixed algebraic closure $\overline{K}$ of K, let $\mathfrak{m} \subset \mathcal{O}_C$ be the maximal ideal in the valuation ring of C, and let $\overline{k}$ be the residue field of $\mathcal{O}_C$.

- We let $\mathfrak{X}$ denote the formal p-adic completion $\widehat{\mathcal{X}_{\mathcal{O}_C}}$ of $\mathcal{X}_{\mathcal{O}_C}$ and let X^{ad} denote the adic generic fiber of $\mathfrak{X}$. There is an open immersion

$$X^{\mathrm{ad}} \hookrightarrow (\mathcal{X}_C)^{\mathrm{ad}} \qquad \text{into the adic space} \quad (\mathcal{X}_C)^{\mathrm{ad}} \quad \text{associated to} \quad \mathcal{X}_C$$

  that is an isomorphism if $\mathcal{X}$ is $\mathcal{O}$-proper (cf. [Con99, 5.3.1 3.–4.], [Hub94, 4.6 (i)], and [Hub96, 1.9.2 ii)]). In particular, X^{ad} is C-smooth (cf. [Con99, 5.2.1 1.] and [Hub96, 1.7.11 ii)]), and X^{ad} is C-proper whenever $\mathcal{X}$ is $\mathcal{O}$-proper (cf. [Hub96, 1.3.18 ii)]).

- We let $X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}$ denote the proétale site of X^{ad} (reviewed in [BMS16, §5.1] and defined in [Sch13, 3.9] and [Sch13e, (1)]) and let

$$\nu \colon X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}} \tag{$\star$}$$

  be the morphism towards the étale site of $\mathfrak{X}$ that sends a variable étale $\mathfrak{U} \to \mathfrak{X}$ to the constant pro-system associated to its adic generic fiber (cf. [Hub96, 3.5.1] and [SP, 00X6]).

We use the étale site of $\mathfrak{X}$ instead of the Zariski site used in [BMS16] because the description ($\square$) is étale (rather than Zariski) local on $\mathcal{X}$.

1.2. Conventions and additional notation. We let $\widehat{M}$ denote the (p-adic, unless specified otherwise) completion of a module M and, similarly, let $\widehat{\bigoplus}$ denote the completion of a direct sum. By default, we endow p-adically complete modules with the inverse limit of discrete topologies. When M is an object of a derived category, we let $\widehat{M} := R\lim(M \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z})$ denote its derived p-adic completion (cf. [SP, 0940] and [BS15, §3]). For power series rings, we use $\{-\}$ to indicate decaying coefficients.

For a ring object R of a topos $\mathscr{T}$, we write $D(\mathscr{T}, R)$ or simply $D(R)$ for the derived category of R-modules; we abuse notation and sometimes also write $D(\mathscr{T}, R)$ when $\mathscr{T}$ is a site. We use the fact that for a morphism of ringed topoi, the functor Rf_* commutes with the formation of derived limits and derived (p-adic, unless specified otherwise) completions, cf. [SP, 0A07 and 0944].

We use the definition [BMS16, 3.5] of a perfectoid ring (the compatibility with prior definitions is discussed in [BMS16, 3.20]). In particular, by [BMS16, 3.9 and 3.10], a p-torsion free ring S is *perfectoid* if and only if S is p-adically complete, S contains a p-power root ϖ of up for some $u \in S^\times$, and the p-power map $S/\varpi S \to S/pS$ is an isomorphism.

We let $W(-)$ denote p-typical Witt vectors and let $[-]$ denote Teichmüller representatives. We let μ_{p^n} be the $\mathbb{Z}$-group scheme that parametrizes $(p^n)^{\text{th}}$ roots of unity and we denote a primitive $(p^n)^{\text{th}}$ root of unity by ζ_{p^n}. We let $(-)^{\text{sm}}$ denote the smooth locus of a (formal) scheme over an implicit base. For a topological ring R, we let R° denote the subset of powerbounded elements.

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2. The object $\widetilde{\Omega}_{\mathfrak{X}}$ and its cohomology

The goal of this section is to analyze an auxiliary object $\widetilde{\Omega}_{\mathfrak{X}}$ of the derived category of $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$-modules; this object will subsequently be key for specializing A_{inf}-valued cohomology to logarithmic de Rham cohomology. We loosely follow [BMS16, §8], the key input to the étale local analysis of $\widetilde{\Omega}_{\mathfrak{X}}$ being the almost purity theorem applied to a suitable perfectoid cover constructed with the help of a coordinate morphism (▫). In comparison to the smooth case, there are two key complications.

Firstly, due to the structure of the singularities on the special fiber of $\mathcal{X}$, the intermediate layers in the étale local perfectoid cover are no longer $\mathfrak{X}$-flat, which complicates transfer of continuous cohomology computations across the map (▫) (contrast this with [BMS16, proof of Prop. 8.9]). We bypass this issue by using results of Gabber–Ramero from [GR03] to ensure that the formal p-adic completion of the (non-Noetherian) $\mathcal{O}_C$-base change of the map (▫) is still flat.

Secondly, the local comparison of $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ with Kähler differentials only applies on the smooth locus. To extend it to the singular locus, we assume that $\mathcal{X}$ is proper (which is entirely sufficient for our later purposes) and proceed indirectly. Namely, in the proof of Theorem 2.22 we first use non-Noetherian formal GAGA due to K. Fujiwara and F. Kato to algebraize a suitable comparison map, we then use results from [Sch13] to analyze this map after inverting p, and we conclude by bootstrapping from the smooth locus, which is possible because $\mathcal{X}$ is (S_2) and $\mathcal{X}\backslash\mathcal{X}^{\text{sm}}$ is of codimension $\geqslant 2$ in $\mathcal{X}$.

We begin by defining the object $\widetilde{\Omega}_{\mathfrak{X}}$ that we are going to study.

2.1. The object $\widetilde{\Omega}_{\mathfrak{X}}$. Throughout §2 we ring the étale site $\mathfrak{X}_{\text{ét}}$ with the structure sheaf $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$ and the proétale site $X^{\text{ad}}_{\text{proét}}$ with the integral completed structure sheaf $\widehat{\mathcal{O}}^+_X := \varprojlim_n (\mathcal{O}^+_{X^{\text{ad}}_{\text{proét}}}/p^n)$ of [Sch13, 4.1], so that, thanks to [Hub96, 1.9.1 b)], the morphism of topoi (ν^{-1}, ν_*) induced by $(\star)$ canonically extends to a morphism of ringed topoi (cf. [SP, 00XC and 01D3]). We set

$$\widetilde{\Omega}_{\mathfrak{X}} := L\eta_{(\zeta_p - 1)}(R\nu_*(\widehat{\mathcal{O}}^+_X)) \in D^{\geqslant 0}(\mathcal{O}_{\mathfrak{X},\,\text{ét}}),$$

where the décalage functor $L\eta$ of [BMS16, §6] is taken with respect to the ideal sheaf $(\zeta_p - 1)\mathcal{O}_{\mathfrak{X},\,\text{ét}}$.

We perform the initial analysis of the cohomology of $\widetilde{\Omega}_{\mathfrak{X}}$ in the proof of the following Proposition 2.2.

Proposition 2.2. *If the relative dimension d of $\mathcal{X}$ over $\mathcal{O}$ is constant, then for every $i \in \mathbb{Z}_{\geqslant 0}$ the $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$-module $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is étale locally on $\mathfrak{X}$ free of rank $\binom{d}{i}$ and, in particular, $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is adically quasi-coherent in the sense that for every $n \geqslant 1$, the $\mathcal{O}_{\mathfrak{X},\,\text{ét}}/p^n$-module $H^i(\widetilde{\Omega}_{\mathfrak{X}})/p^n$ is quasi-coherent.*

The claim of Proposition 2.2 is étale local, so we assume until Corollary 2.15 (i.e., until the end of the proof of Proposition 2.2) that $\mathfrak{X} = \operatorname{Spf} R$, that $\mathfrak{X}$ is connected, and that there is an étale

Spf $\mathcal{O}_C$-morphism

$$\mathfrak{X} = \operatorname{Spf} R \to \operatorname{Spf} R^{\square} \qquad \text{with} \qquad R^{\square} := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - \pi) \tag{2.2.1}$$

for some $0 \leqslant r \leqslant d$ and uniformizer $\pi \in \mathcal{O}$. By [GR03, 7.1.6 (i)] (with [SP, 04D1]), R is $R^{\square}$-flat.

We now use (2.2.1) to build a suitable perfectoid cover of R.

2.3. The perfectoid R_∞. We fix compatible p-power roots $(\dots, \pi^{1/p}, \pi)$ of π in $\mathcal{O}_C$ and set

$$R^{\square}_m := \mathcal{O}_C\{t_0^{1/p^m}, \dots, t_r^{1/p^m}, t_{r+1}^{\pm 1/p^m}, \dots, t_d^{\pm 1/p^m}\}/(t_0^{1/p^m} \cdots t_r^{1/p^m} - \pi^{1/p^m}) \quad \text{for} \quad m \in \mathbb{Z}_{\geqslant 0}.$$

The p-adically completed direct sum decomposition

$$R^{\square}_\infty := \widehat{\varinjlim R^{\square}_m} = \widehat{\bigoplus}_{\substack{(a_0, \dots, a_d) \in \mathbb{Z}[\frac{1}{p}]_{\geqslant 0}^{\oplus(r+1)} \oplus \mathbb{Z}[\frac{1}{p}]^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d} \tag{2.3.1}$$

shows that for $m \geqslant 0$ the map $R^{\square}_m \to R^{\square}_\infty$ is an inclusion of an $R^{\square}_m$-module direct summand consisting of the summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of (2.3.1) with $p^m a_j \in \mathbb{Z}$ for all j, and that $R^{\square}_\infty$ is perfectoid (cf. §1.2). We set

$$R_m := R \otimes_{R^{\square}} R^{\square}_m \quad \text{for} \quad m \in \mathbb{Z}_{\geqslant 0} \qquad \text{and} \qquad R_\infty := \widehat{\varinjlim R_m} \cong (R \otimes_{R^{\square}} R^{\square}_\infty)\widehat{\ }.$$

The rings R_m and R_∞ are p-torsion free and, by [GR03, 7.1.6 (ii)], each R_m is p-adically complete. The ring R_∞ is perfectoid because so is $R^{\square}_\infty$.

2.4. The tower X^{ad}_∞. The profinite group

$$\Delta := \left\{ (\epsilon_0, \dots, \epsilon_d) \in \left(\varprojlim_{m \geqslant 0} \mu_{p^m}(\mathcal{O}_C) \right)^{\oplus(d+1)} \,\middle|\, \epsilon_0 \cdots \epsilon_r = 1 \right\} \simeq \mathbb{Z}_p^{\oplus d}$$

acts $R^{\square}$-linearly on $R^{\square}_m$ by scaling each t_j^{1/p^m} by the μ_{p^m}-component of ϵ_j. The induced action of Δ on $R^{\square}_\infty$ is continuous and preserves the decomposition (2.3.1). Since

$$R^{\square}_m[\tfrac{1}{p}] = \bigoplus_{a_1, \dots, a_d \in \{0, \frac{1}{p^m}, \dots, \frac{p^m - 1}{p^m}\}} R^{\square}[\tfrac{1}{p}] \cdot t_1^{a_1} \cdots t_d^{a_d} \qquad \text{for each } m \geqslant 0,$$

$R^{\square}_m[\frac{1}{p}]$ identifies with the $R^{\square}[\frac{1}{p}]$-algebra obtained by adjoining $(p^m)^{\text{th}}$ roots of $t_1, \dots, t_d \in R^{\square}[\frac{1}{p}]^\times$, so is finite étale over $R^{\square}[\frac{1}{p}]$. Therefore, $\varinjlim_m R^{\square}_m[\frac{1}{p}]$ is a pro-(finite étale Galois) Δ-cover of $R^{\square}[\frac{1}{p}]$. The reducedness of the closed $\mathcal{O}_C$-fiber of $R^{\square}_m$ ensures that $R^{\square}_m = (R^{\square}_m[\frac{1}{p}])^\circ$, so the pro-object

$$X^{\square}_\infty := \text{“} \varprojlim \operatorname{Spa}(R^{\square}_m[\tfrac{1}{p}], R^{\square}_m) \text{”}$$

is an affinoid perfectoid pro-(finite étale Galois) Δ-cover of the adic generic fiber $X^{\square}$ of $\operatorname{Spf} R^{\square}$.

Since R is the p-adic completion of an étale $R^{\square}$-algebra, $X^{\mathrm{ad}} = \operatorname{Spa}(R[\frac{1}{p}], R)$, so, in particular, X^{ad} inherits connectedness from $\operatorname{Spf} R$. It then follows that the tower

$$X^{\mathrm{ad}}_\infty := \text{“} \varprojlim \operatorname{Spa}(R_m[\tfrac{1}{p}], R_m), \text{”}$$

which is the X^{ad}-base change of $X^{\square}_\infty$, is an affinoid perfectoid pro-(finite étale Galois) Δ-cover of X^{ad}.

We now use the tower X^{ad}_∞ to relate $H^i(X^{\mathrm{ad}}_{\text{proét}}, \widehat{\mathcal{O}}^+_X)$ to continuous group cohomology of Δ.

2.5. The almost purity input. By [Sch13, 3.5, 3.7 (iii) and its proof, 4.10 (iii) 6.6], the i^{th} Čech cohomology group with $\widehat{\mathcal{O}}^+_X$-coefficients (resp., $\widehat{\mathcal{O}}^+_{X^{\square}}$-coefficients) of the iterated fiber product hypercover that results from $X^{\mathrm{ad}}_\infty \twoheadrightarrow X^{\mathrm{ad}}$ (resp., $X^{\square}_\infty \twoheadrightarrow X^{\square}$) identifies with the continuous profinite group cohomology $H^i_{\text{cont}}(\Delta, R_\infty)$ (resp., $H^i_{\text{cont}}(\Delta, R^{\square}_\infty)$). In particular, the Cartan–Leray spectral sequence of [SGA 4_{II}, V.3.3] (i.e., the spectral sequence of a hypercover, cf. [SP, 01GY]) supplies the edge homomorphism

$$e\colon H^i_{\text{cont}}(\Delta, R_\infty) \to H^i(X^{\text{ad}}_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \qquad (\text{resp.,}\quad e^\square\colon H^i_{\text{cont}}(\Delta, R^\square_\infty) \to H^i(X^\square_{\text{proét}}, \widehat{\mathcal{O}}^+_{X^\square})). \quad (2.5.1)$$

By the almost purity theorem [Sch13, 4.10 (v)], the maximal ideal $\mathfrak{m} \subset \mathcal{O}_C$ kills both $\operatorname{Ker} e$ and $\operatorname{Coker} e$, and likewise for $e^\square$.

Our analysis of $H^i_{\text{cont}}(\Delta, R_\infty)$ and $H^i_{\text{cont}}(\Delta, R^\square_\infty)$ will use the following four lemmas.

Lemma 2.6. *Suppose that $i \in \mathbb{Z}_{\geqslant 0}$, that H is a profinite group, that $\{M_j\}_{j\in J}$ are p-adically complete, p-torsion free, continuous H-modules, and that either*

(i) the group $H^i_{\text{cont}}(H, M_j)$ is p-torsion free for every j, or

(ii) there is an $n \in \mathbb{Z}_{\geqslant 0}$ such that p^n kills $H^i_{\text{cont}}(H, M_j)$ for every j.

Then the following map is injective:

$$H^i_{\text{cont}}(H, \widehat{\textstyle\bigoplus}_{j\in J} M_j) \hookrightarrow \textstyle\prod_{j\in J} H^i_{\text{cont}}(H, M_j), \qquad \textit{where the completion is } p\textit{-adic.}$$

In particular, in the case (i) (resp., (ii)), $H^i_{\text{cont}}(H, \widehat{\bigoplus}_{j\in J} M_j)$ is p-torsion free (resp., killed by p^n).

Proof. Let c be a continuous $\left(\widehat{\bigoplus}_{j\in J} M_j\right)$-valued i-cocycle that represents an element of the kernel. For each j, let c_j be the "j^{th}-coordinate" of c. We discard j with $c_j = 0$ and for each remaining j we choose the maximal $n_j \in \mathbb{Z}_{\geqslant 0}$ for which c_j is $p^{n_j} M_j$-valued, so that $n_j \to \infty$. Since M_j is p-torsion free, $p^{-n_j} c_j$ is an M_j-valued continuous i-cocycle.

In the case (i), the class of $p^{-n_j} c_j$ in $H^i_{\text{cont}}(H, M_j)$ vanishes, so c_j is the coboundary of a $p^{n_j} M_j$-valued continuous $(i-1)$-cochain b_j. In the case (ii), p^n kills $H^i_{\text{cont}}(H, M_j)$, so c_j is the coboundary of a $p^{n_j - n} M_j$-valued continuous $(i-1)$-cochain b_j whenever $n_j \geqslant n$.

In both cases, the b_j exhibit c as a continuous coboundary. ☐

Lemma 2.7 ([BMS16, 7.3 (ii)]). *If H is a profinite group isomorphic to $\mathbb{Z}_p^{\oplus d}$ for some $d \in \mathbb{Z}_{>0}$ and M is a p-adically complete continuous H-module, then for any topological generators $\gamma_1, \ldots, \gamma_d$ of H the groups $H^*_{\text{cont}}(H, M)$ identify with the cohomology groups of the Koszul complex*

$$M \otimes_{\mathbb{Z}[f_1,\ldots,f_d]} \textstyle\bigotimes_{i=1}^d \left(\mathbb{Z}[f_1, \ldots, f_d] \xrightarrow{f_i} \mathbb{Z}[f_1, \ldots, f_d]\right)$$

in which the factor complexes are concentrated in degrees 0 and 1, the tensor products are over $\mathbb{Z}[f_1, \ldots, f_d]$, and M is regarded as a $\mathbb{Z}[f_1, \ldots, f_d]$-module by letting f_j act as $\gamma_j - 1$. ☐

Lemma 2.8 ([BMS16, 8.11 (i)]). *An $\mathcal{O}_C$-module map $f\colon M \to M'$ for which $\mathfrak{m}$ kills both* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *and $M[\mathfrak{m}] = \left(\frac{M}{(\zeta_p - 1)M}\right)[\mathfrak{m}] = 0$ induces an isomorphism $\frac{M}{M[\zeta_p-1]} \xrightarrow{\sim} \frac{M'}{M'[\zeta_p-1]}$.* ☐

Lemma 2.9. *For an inclusion $\mathfrak{o} \subset \mathfrak{O}$ of a discrete valuation ring into a nondiscrete valuation ring of rank 1, if N is an $\mathfrak{o}$-module and $\mathfrak{M} \subset \mathfrak{O}$ denotes the maximal ideal, then $(N \otimes_{\mathfrak{o}} \mathfrak{O})[\mathfrak{M}] = 0$.*

Proof. The $\mathfrak{o}$-flatness of $\mathfrak{O}$ reduces us to the case when N is finitely generated, so it suffices to observe that $(\mathfrak{O}/(a))[\mathfrak{M}] = 0$ whenever $a \in \mathfrak{O}$. ☐

The following lemma and its subsequent corollary are steps towards the proof of Proposition 2.2.

Lemma 2.10. *Let $M^{\square}_{\infty}$ denote the p-adically completed direct sum of those summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of* (2.3.1) *for which $a_j \notin \mathbb{Z}$ for some j, so that we have the Δ-decomposition $R^{\square}_{\infty} \cong R^{\square} \oplus M^{\square}_{\infty}$. The $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, M^{\square}_{\infty})$ is killed by $\zeta_p - 1$ and has no nonzero $\mathfrak{m}$-torsion.*

Proof. By Lemma 2.7, $H^i_{\mathrm{cont}}(\Delta, \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d})$ identifies with the i^{th} cohomology group of the $\mathcal{O}_C$-tensor product of d complexes of the form $\mathcal{O}_C \xrightarrow{\zeta - 1} \mathcal{O}_C$ for suitable p-power roots of unity ζ. If $a_j \notin \mathbb{Z}$ for some j, then some ζ is not 1, and the corresponding factor complex is quasi-isomorphic to $\mathcal{O}_C/(\zeta - 1)$. Thus, $\zeta - 1$, and hence also $\zeta_p - 1$, kills each such $H^i_{\mathrm{cont}}(\Delta, \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d})$. Moreover, since the d complexes may be defined over some discrete valuation subring of $\mathcal{O}_C$, Lemma 2.9 ensures that $H^i_{\mathrm{cont}}(\Delta, \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d})[\mathfrak{m}] = 0$. The desired claim now follows from Lemma 2.6. $\square$

Corollary 2.11. *The map $e^{\square}$ of* (2.5.1) *induces an isomorphism*

$$\frac{H^i_{\mathrm{cont}}(\Delta, R^{\square}_{\infty})}{H^i_{\mathrm{cont}}(\Delta, R^{\square}_{\infty})[\zeta_p - 1]} \xrightarrow{\sim} \frac{H^i(X^{\square}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X^{\square}})}{H^i(X^{\square}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X^{\square}})[\zeta_p - 1]} \tag{2.11.1}$$

whose source and target are free $R^{\square}$-modules of rank $\binom{d}{i}$.

Proof. By Lemma 2.7, the continuous group cohomology of Δ acting (trivially) on $R^{\square}$ identifies with the cohomology of the complex $\bigotimes_{i=1}^{d} \left(R^{\square} \xrightarrow{0} R^{\square} \right)$. Therefore,

$$H^i_{\mathrm{cont}}(\Delta, R^{\square}) \cong (R^{\square})^{\oplus \binom{d}{i}}, \tag{2.11.2}$$

so, by Lemma 2.10, the source of (2.11.1) is free of rank $\binom{d}{i}$. Since $\mathfrak{m}$ kills $\mathrm{Ker}(e^{\square})$ and $\mathrm{Coker}(e^{\square})$ (see §2.5), the desired conclusion follows by combining Lemma 2.8 with Lemma 2.10 and (2.11.2). $\square$

We keep the notation $M^{\square}_{\infty}$ and proceed to the analogues of Lemma 2.10 and Corollary 2.11 for R_{∞}.

Lemma 2.12. *For $m \in \mathbb{Z}_{>0}$, let $M^{\square}_m$ denote the p-adically completed direct sum of those summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of* (2.3.1) *for which m is the smallest positive integer with $p^m \cdot (a_0, \ldots, a_d) \in \mathbb{Z}^{\oplus(d+1)}$.*

(a) *The $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, M^{\square}_m \otimes_{R^{\square}} R)$ is killed by $\zeta_p - 1$ and has no nonzero $\mathfrak{m}$-torsion.*

(b) *The $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, M^{\square}_{\infty} \widehat{\otimes}_{R^{\square}} R)$ is killed by $\zeta_p - 1$ and has no nonzero $\mathfrak{m}$-torsion.*

Proof. By [GR03, 7.1.6], R is $R^{\square}$-flat and $M^{\square}_m \otimes_{R^{\square}} R$ is p-adically complete. Therefore, by Lemma 2.7,

$$H^i_{\mathrm{cont}}(\Delta, M^{\square}_m \otimes_{R^{\square}} R) \cong H^i_{\mathrm{cont}}(\Delta, M^{\square}_m) \otimes_{R^{\square}} R,$$

so that, by Lemma 2.10, $(\zeta_p - 1) \cdot H^i_{\mathrm{cont}}(\Delta, M^{\square}_m \otimes_{R^{\square}} R) = 0$. Since $M^{\square}_m \otimes_{R^{\square}} R$ is p-torsion free, it follows that the map $H^i_{\mathrm{cont}}(\Delta, p(M^{\square}_m \otimes_{R^{\square}} R)) \to H^i_{\mathrm{cont}}(\Delta, M^{\square}_m \otimes_{R^{\square}} R)$ vanishes. Therefore, the map

$$H^i_{\mathrm{cont}}(\Delta, M^{\square}_m \otimes_{R^{\square}} R) \to H^i_{\mathrm{cont}}(\Delta, (M^{\square}_m \otimes_{R^{\square}} R)/p) \overset{2.7}{\cong} H^i_{\mathrm{cont}}(\Delta, M^{\square}_m/p) \otimes_{R^{\square}/p} R/p \tag{2.12.1}$$

is injective. The finitely presented $\mathcal{O}_C$-algebra $R^{\square}/p$, its finite module $R^{\square}_m/p$, the Δ-action on $R^{\square}_m/p$, and the finitely presented $R^{\square}/p$-algebra R/p can all be defined over a suitably large discrete valuation subring of $\mathcal{O}_C$. Therefore, by Lemmas 2.7 and 2.9, $H^i_{\mathrm{cont}}(\Delta, R^{\square}_m/p) \otimes_{R^{\square}/p} R/p$ has no nonzero $\mathfrak{m}$-torsion, and the same holds for its direct summand $H^i_{\mathrm{cont}}(\Delta, M^{\square}_m/p) \otimes_{R^{\square}/p} R/p$. Together with (2.12.1), this proves (a). Since $M^{\square}_{\infty} \widehat{\otimes}_{R^{\square}} R \cong \widehat{\bigoplus} (M^{\square}_m \otimes_{R^{\square}} R)$, (b) follows from (a) and Lemma 2.6. $\square$

Corollary 2.13. *The map e of* (2.5.1) *induces an isomorphism*

$$\frac{H^i_{\mathrm{cont}}(\Delta, R_{\infty})}{H^i_{\mathrm{cont}}(\Delta, R_{\infty})[\zeta_p - 1]} \xrightarrow{\sim} \frac{H^i(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X)}{H^i(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X)[\zeta_p - 1]}, \tag{2.13.1}$$

whose source and target are free R-modules of rank $\binom{d}{i}$.

Proof. Since R is $R^\square$-flat, Lemma 2.7 implies that

$$R^{\oplus\binom{d}{i}} \overset{(2.11.2)}{\cong} H^i_{\mathrm{cont}}(\Delta, R^\square)\otimes_{R^\square} R \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, R).$$

It remains to combine the Δ-decomposition $R_\infty \cong R \oplus (M^\square_\infty \widehat{\otimes}_{R^\square} R)$ with Lemmas 2.8 and 2.12 (b). ☐

With the preparations above, we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. Since the maps $e^\square$ and e of (2.5.1) are compatible, Corollaries 2.11 and 2.13 and their proofs show that the base change morphism

$$\frac{H^i(X^\square_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X^\square})}{H^i(X^\square_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X^\square})[\zeta_p - 1]} \otimes_{R^\square} R \to \frac{H^i(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X})}{H^i(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X})[\zeta_p - 1]} \tag{2.13.2}$$

is an isomorphism of free R-modules of rank $\binom{d}{i}$. Since the connected affine $\mathfrak{X}$ is arbitrary (subject to (2.2.1)), we conclude that

$$\frac{H^i(X^\square_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X^\square})}{H^i(X^\square_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X^\square})[\zeta_p - 1]} \otimes_{R^\square} \mathcal{O}_{\mathrm{Spf}\, R,\, \mathrm{\acute{e}t}} \xrightarrow{\sim} \frac{R^i\nu_*(\widehat{\mathcal{O}}^+_X)}{(R^i\nu_*(\widehat{\mathcal{O}}^+_X))[\zeta_p - 1]} \overset{\text{[BMS16, 6.4]}}{\cong} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \tag{2.13.3}$$

and that $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is free of rank $\binom{d}{i}$, as desired. ☐

Remark 2.14. The proof of Proposition 2.2, specifically, (2.13.2) and (2.13.3) applied to a variable $\mathfrak{X}$, shows that if $\mathfrak{X}$ is affine, connected, and admits a coordinate map (2.2.1), then the presheaf which to a variable $\mathfrak{X}$-étale affine $\mathfrak{X}'$ assigns $\frac{H^i(X'^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X'})}{H^i(X'^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X'})[\zeta_p - 1]}$ is already a sheaf.

Having completed the proof of Proposition 2.2, we no longer assume (2.2.1). However, whenever we refer to (2.2.1), we also employ its associated notational apparatus established above.

Corollary 2.15. *The map $\nu^\sharp\colon \mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}} \to \nu_*\widehat{\mathcal{O}}^+_X$ is an isomorphism. In particular,*

$$H^0(\widetilde{\Omega}_{\mathfrak{X}}) \cong \mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}. \tag{2.15.1}$$

Proof. Since $(\nu_*\widehat{\mathcal{O}}^+_X)[\zeta_p - 1] = 0$, (2.15.1) follows from the first claim. For the latter, we may work étale locally on $\mathfrak{X}$, so we put ourselves in the situation (2.2.1) and, due to the discussion in §2.5, we need to show that the map $R \hookrightarrow (R_\infty)^\Delta$ is an isomorphism. Its cokernel, $(M^\square_\infty \widehat{\otimes}_{R^\square} R)^\Delta$, is both p-torsion free and, by Lemma 2.12 (b), killed by $\zeta_p - 1$. ☐

Our next task is to relate $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ to $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ for $i > 0$.

2.16. Cup products. By [SP, 0B6C],[1] there is a cup product map

$$R\nu_*(\widehat{\mathcal{O}}^+_X) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}} R\nu_*(\widehat{\mathcal{O}}^+_X) \to R\nu_*(\widehat{\mathcal{O}}^+_X). \tag{2.16.1}$$

Moreover, arguments analogous to those used to construct the map [SP, 068H] give product maps

$$R^j\nu_*(\widehat{\mathcal{O}}^+_X) \otimes_{\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}} R^{j'}\nu_*(\widehat{\mathcal{O}}^+_X) \xrightarrow{-\cup-} H^{j+j'}(R\nu_*(\widehat{\mathcal{O}}^+_X) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}} R\nu_*(\widehat{\mathcal{O}}^+_X)), \tag{2.16.2}$$

which satisfy $x \cup y = (-1)^{jj'} y \cup x$ (cf. [SP, 0BYI]) and combine with (2.16.1) to give the map $\bigotimes_{s=1}^i R^1\nu_*(\widehat{\mathcal{O}}^+_X) \to R^i\nu_*(\widehat{\mathcal{O}}^+_X)$ for $i \in \mathbb{Z}_{>0}$. Thus, since, by Proposition 2.2, $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ has no nontrivial 2-torsion, this latter map gives rise to the $\mathcal{O}_{\mathfrak{X},\, \mathrm{\acute{e}t}}$-module map

$$\bigwedge^i \left(\frac{R^1\nu_*(\widehat{\mathcal{O}}^+_X)}{R^1\nu_*(\widehat{\mathcal{O}}^+_X)[\zeta_p - 1]} \right) \cong \bigwedge^i H^1(\widetilde{\Omega}_{\mathfrak{X}}) \to H^i(\widetilde{\Omega}_{\mathfrak{X}}) \cong \frac{R^i\nu_*(\widehat{\mathcal{O}}^+_X)}{R^i\nu_*(\widehat{\mathcal{O}}^+_X)[\zeta_p - 1]}. \tag{2.16.3}$$

[1] Loc. cit. applies in its present form because $X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}$ has enough points by [Sch13e, (2)].

Proposition 2.17. *For each $i \in \mathbb{Z}_{>0}$, the map* (2.16.3) *is an isomorphism.*

Proof. We may work étale locally, so we put ourselves in the situation (2.2.1). The edge maps

$$e\colon H^i_{\mathrm{cont}}(\Delta, R_\infty) \to H^i(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X)$$

of (2.5.1) are compatible with cup products: in order to check this one identifies $H^i(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X)$ with the direct limit of the i^{th} Čech cohomology groups of $\widehat{\mathcal{O}}^+_X$ with respect to a variable proétale hypercovering of X^{ad} (cf. [SP, 01H0]) and uses the hypercovering construction of the cup product (cf. [SP, 01FP]). Due to Corollaries 2.11 and 2.13 and their proofs, it remains to argue that

$$H^1_{\mathrm{cont}}(\Delta, R^\square) \overset{(2.11.2)}{\cong} (R^\square)^d \qquad \text{induces} \qquad H^i_{\mathrm{cont}}(\Delta, R^\square) \overset{(2.11.2)}{\cong} \textstyle\bigwedge^i (R^\square)^d$$

via cup product, which follows from [BMS16, 7.3 and 7.5]. □

Corollary 2.15 and Proposition 2.17 reduce the task of explicating $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ to the case $i = 1$. We will relate $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ to differential forms on $\mathfrak{X}$ under a simplifying properness assumption (see Theorem 2.22), and for this we need the preparations of §§2.18–2.21. While §2.18 reviews material that was also used in [BMS16, §8.2], §§2.19–2.21 slightly differ from loc. cit. because of the possible nonsmoothness of our $\mathcal{X}$.

2.18. The completed cotangent complex $\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p}$. By a result of Colmez [Sch13, 4.7], affinoid perfectoids form a basis of the topology of $X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}$. Therefore, [BMS16, 3.14] ensures that the cotangent complex $\mathbb{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_C} \in D^{\leqslant 0}(\widehat{\mathcal{O}}^+_X)$ (whose levelwise terms are $\widehat{\mathcal{O}}^+_X$-flat) satisfies

$$\mathbb{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_C} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong 0 \qquad \text{and hence also} \qquad \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_C} \cong 0.$$

Consequently, the derived p-adic completion of the canonical morphism

$$\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X \to \mathbb{L}_{\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p}$$

is an isomorphism. Moreover, by [GR03, 6.5.12 (ii)], $\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p}$ is quasi-isomorphic to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ placed in degree 0. The p-divisibility of $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ then ensures that

$$\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes^{\mathbb{L}}_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X/p^n\widehat{\mathcal{O}}^+_X \cong (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n] \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X)[1] \overset{\text{[Sch13, 4.2 (iii)]}}{\cong} (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n] \otimes_{\mathcal{O}_C} \mathcal{O}^+_X/p^n\mathcal{O}^+_X)[1].$$

Since, by [Fon82, Thm. 1′ (ii)],[2] $\mathcal{O}_C\{1\} := \varprojlim_n \Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n]$ is a free $\mathcal{O}_C$-module of rank 1, it follows that the derived p-adic completion of $\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X$ identifies with $\widehat{\mathcal{O}}^+_X\{1\}$ placed in degree -1, where $\{1\}$ abbreviates the $\mathcal{O}_C$-tensor product with $\mathcal{O}_C\{1\}$. In conclusion, we obtain an isomorphism

$$\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} \cong \widehat{\mathcal{O}}^+_X\{1\}[1] \quad \text{in} \quad D(\widehat{\mathcal{O}}^+_X). \tag{2.18.1}$$

2.19. The completed cotangent complex $\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}/\mathbb{Z}_p}$. By an argument analogous to the one used in §2.18, the derived p-adic completion of $\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}$ identifies with $\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}\{1\}$ placed in degree -1. This gives the first isomorphism in

$$H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}/\mathbb{Z}_p}) \cong H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\mathrm{\acute{e}t}}/\mathcal{O}_C}) \overset{\text{[GR03, 7.2.4, 7.2.8]}}{\cong} \widehat{\Omega}^1_{\mathcal{X}}, \tag{2.19.1}$$

where $\widehat{\Omega}^1_{\mathcal{X}}$ denotes the formal p-adic completion of $\Omega^1_{\mathcal{X}_{\mathcal{O}_C}/\mathcal{O}_C}$. Moreover, by [GR03, 7.2.10 (iii)], the second isomorphism of (2.19.1) induces a quasi-isomorphism $\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}^{\mathrm{sm}},\,\mathrm{\acute{e}t}}/\mathcal{O}_C} \cong (\widehat{\Omega}^1_{\mathcal{X}}|_{\mathfrak{X}^{\mathrm{sm}}})[0]$.

[2] For passage from $\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}$ of loc. cit. to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$, one may use [GR03, 6.5.20 (i)] to get that $\Omega^1_{\mathcal{O}_C/\overline{\mathbb{Z}}_p}[p] = 0$.

2.20. The map $\widehat{\Omega}^1_{\mathcal{X}}\{-1\} \to R^1\nu_*\widehat{\mathcal{O}}^+_X$. The morphism of ringed topoi discussed in §2.1 gives rise to the pullback morphism $\mathbb{L}_{\mathcal{O}_{\mathfrak{X},\,\text{ét}}/\mathbb{Z}_p} \to R\nu_*\mathbb{L}_{\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p}$, and hence also to

$$\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\text{ét}}/\mathbb{Z}_p} \to R\nu_*\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} \overset{(2.18.1)}{\cong} R\nu_*\widehat{\mathcal{O}}^+_X\{1\}[1], \qquad \text{so to} \qquad \widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\,\text{ét}}/\mathbb{Z}_p}\{-1\}[-1] \to R\nu_*\widehat{\mathcal{O}}^+_X.$$

By applying $H^1(-)$ and using (2.19.1), we arrive at the $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$-module morphism

$$\widehat{\Omega}^1_{\mathcal{X}}\{-1\} \to R^1\nu_*\widehat{\mathcal{O}}^+_X, \tag{2.20.1}$$

which, by [BMS16, proof of Prop. 8.15], is an isomorphism onto $\left((\zeta_p - 1)\cdot R^1\nu_*\widehat{\mathcal{O}}^+_X\right)\Big|_{\mathfrak{X}^{\text{sm}}}$ over $\mathfrak{X}^{\text{sm}}$. This map will facilitate our analysis of $H^1(\widetilde{\Omega}^1_{\mathfrak{X}})$ in Theorem 2.22.

2.21. The sheaf $\Omega^1_{\mathcal{X},\,\log}$. The semistable reduction of $\mathcal{X}$ over $\mathcal{O}$ ensures that if we endow $\mathcal{X}$ (resp., $\operatorname{Spec}\mathcal{O}$) with the log structure that arises from the inclusion of the divisor $\mathcal{X}_k$ (resp., $\operatorname{Spec} k$), then the resulting morphism of log schemes is log smooth, cf. [Kat89, 3.7 (2)]. In particular, the resulting $\mathcal{O}_{\mathcal{X}}$-module $\Omega^1_{\mathcal{X},\,\log}$ of logarithmic differentials is locally free of rank equal to the (locally constant) relative dimension of $\mathcal{X}$ over $\mathcal{O}$. The natural map

$$\Omega^1_{\mathcal{X}/\mathcal{O}} \to \Omega^1_{\mathcal{X},\,\log}$$

is an isomorphism over the fiberwise dense open smooth locus $\mathcal{X}^{\text{sm}}$ of $\mathcal{X}$. For $i \in \mathbb{Z}_{\geqslant 0}$, we set $\Omega^i_{\mathcal{X},\,\log} := \bigwedge^i \Omega^1_{\mathcal{X},\,\log}$, so that $\Omega^i_{\mathcal{X},\,\log}$ is a vector bundle on $\mathcal{X}$ that restricts to $\Omega^i_{\mathcal{X}^{\text{sm}}/\mathcal{O}}$ on $\mathcal{X}^{\text{sm}}$.

Theorem 2.22. *For every $i \in \mathbb{Z}_{\geqslant 0}$, there is an $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$-module isomorphism*

$$H^i(\widetilde{\Omega}_{\mathfrak{X}}) \simeq \widehat{\Omega}^i_{\mathcal{X},\,\log}\{-i\}, \tag{2.22.1}$$

where $\widehat{\Omega}^i_{\mathcal{X},\,\log}$ denotes the formal p-adic completion of the base change of $\Omega^i_{\mathcal{X},\,\log}$ to $\mathcal{X}_{\mathcal{O}_C}$.

In the proof of Theorem 2.22 we will use the formal GAGA and Grothendieck existence theorems. The Noetherian cases of these theorems proved in [EGA III_1, §5] have been extended to suitable non-Noetherian settings by K. Fujiwara and F. Kato (with important inputs due to O. Gabber). The relevant to our aims special case of this extension is summarized in the following theorem.

Theorem 2.23 (Fujiwara–Kato)**.** *For a complete valuation ring V of height 1, a nonzero nonunit $a \in V$, and a proper, finitely presented V-scheme Y, the functor*

$$\mathcal{F} \mapsto (\mathcal{F}/a^n\mathcal{F})_{n\in\mathbb{Z}_{>0}} \tag{2.23.1}$$

from the category of finitely presented $\mathcal{O}_Y$-modules to that of sequences $(\mathcal{F}_n)_{n\in\mathbb{Z}_{>0}}$ of finitely presented $\mathcal{O}_{Y_{V/a^n}}$-modules $\mathcal{F}_n$ equipped with isomorphisms $\mathcal{F}_{n+1}|_{Y_{V/a^n}} \simeq \mathcal{F}_n$ is an equivalence of categories.

Proof. The claim is a special case of [FK14, I.10.1.2]. In order to explain why loc. cit. applies, we first reinterpret our source and target categories.

By a result of Gabber [FK14, 0.9.2.7], V is "a-adically topologically universally adhesive," so, by [FK14, 0.8.5.25 (2)], V is also "topologically universally coherent with respect to (a)." In particular, by [FK14, 0.8.5.24], every finitely presented V-algebra is a coherent ring, and hence, by [FK14, 0.5.1.2], $\mathcal{O}_Y$ is a coherent $\mathcal{O}_Y$-module. In particular, by [FK14, 0.4.1.8], an $\mathcal{O}_Y$-module $\mathcal{F}$ is finitely presented if and only if $\mathcal{F}$ is coherent, and likewise for $\mathcal{O}_{Y_{V/a^n}}$-modules for $n \in \mathbb{Z}_{>0}$.

By [FK14, 0.8.5.19 (3) and 0.8.4.2], the formal a-adic completion $\widehat{Y}$ of Y may be covered by open affines whose coordinate rings are "topologically universally adhesive" and hence, by [FK14, 0.8.5.18], also "topologically universally Noetherian outside (a)." In particular, by [FK14, I.2.1.7 and I.2.1.1 (1)],

$\widehat{Y}$ is "universally rigid-Noetherian." In addition, by [FK14, 0.8.4.5], $\widehat{Y}$ is locally of finite presentation over $\operatorname{Spf} V$, so [FK14, I.7.2.2] applied with $A = V$ and [FK14, I.7.2.1] imply that $\widehat{Y}$ is "universally cohesive." Then, by [FK14, I.7.2.4 and I.3.4.1], the functor $(\mathcal{F}_n) \mapsto \varprojlim \mathcal{F}_n$ is an equivalence from the target category of (2.23.1) to the category of coherent $\mathcal{O}_{\widehat{Y}}$-modules.

In conclusion, our claim is that the quasi-coherent pullback i^* along the morphism $i \colon \widehat{Y} \to Y$ of locally ringed spaces induces an equivalence between the category of coherent $\mathcal{O}_Y$-modules and that of coherent $\mathcal{O}_{\widehat{Y}}$-modules. This is a special case of [FK14, I.10.1.2] (cf. also [FK14, I.§9.1]). □

Remarks.

2.24. In Theorem 2.23, if each $\mathcal{F}_n$ is locally free, then the $\mathcal{O}_Y$-module $\mathcal{F}$ that algebraizes the sequence $(\mathcal{F}_n)$ is also locally free. Indeed, it is enough to argue that the stalks of $\mathcal{F}$ at the points of $Y_{V/a}$ are flat, so, since, by [FK14, I.1.4.7 (2)], the morphism i is flat, it suffices to note that the $\mathcal{O}_{\widehat{Y}}$-module $i^*\mathcal{F} \cong \varprojlim \mathcal{F}_n$ is locally free because the Nakayama lemma ensures that $\mathcal{F}_{n+1}$ is locally trivialized by any lifts of local sections that trivialize $\mathcal{F}_n$.

2.25. Remark 2.24 and the proof of Theorem 2.23 also show that i is flat and that the functor $(\mathcal{F}_n) \mapsto \varprojlim \mathcal{F}_n$ is an equivalence towards the category of finitely presented $\mathcal{O}_{\widehat{Y}}$-modules.

Proof of Theorem 2.22. By Corollary 2.15 and Proposition 2.17, we loose no generality by assuming that $i = 1$. We let $\mathcal{I}$ denote the image of $\Omega^1_{\mathcal{X}/\mathcal{O}} \to \Omega^1_{\mathcal{X},\log}$ and let $\widehat{\mathcal{I}}$ denote the formal p-adic completion of the pullback $\mathcal{I}_{\mathcal{O}_C}$ of $\mathcal{I}$ to $\mathcal{X}_{\mathcal{O}_C}$, so that $\widehat{\Omega}^1_{\mathcal{X}}|_{\mathfrak{X}^{\mathrm{sm}}} \cong \widehat{\mathcal{I}}|_{\mathfrak{X}^{\mathrm{sm}}} \cong \widehat{\Omega}^1_{\mathcal{X},\log}|_{\mathfrak{X}^{\mathrm{sm}}}$.

By Proposition 2.2, $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ is a formal vector bundle, so it has no nonzero local sections that vanish on $\mathfrak{X}^{\mathrm{sm}}$ (cf. [EGA IV$_2$, 6.4.2 and 5.7.5]). In particular, the composite

$$\widehat{\Omega}^1_{\mathcal{X}}\{-1\} \xrightarrow{(2.20.1)} R^1\nu_*\widehat{\mathcal{O}}^+_X \to \frac{R^1\nu_*\widehat{\mathcal{O}}^+_X}{(R^1\nu_*\widehat{\mathcal{O}}^+_X)[\zeta_p - 1]} \cong H^1(\widetilde{\Omega}_{\mathfrak{X}}) \tag{2.25.2}$$

factors through $\widehat{\mathcal{I}}\{-1\}$. Moreover, as we discussed in §2.20, over $\mathfrak{X}^{\mathrm{sm}}$ the induced map

$$\widehat{\mathcal{I}}\{-1\} \to \frac{R^1\nu_*\widehat{\mathcal{O}}^+_X}{(R^1\nu_*\widehat{\mathcal{O}}^+_X)[\zeta_p - 1]} \cong H^1(\widetilde{\Omega}_{\mathfrak{X}}) \qquad \text{factors through} \qquad (\zeta_p - 1)\cdot H^1(\widetilde{\Omega}_{\mathfrak{X}}),$$

so the same holds over entire $\mathfrak{X}$. We will show that the restriction to $\mathfrak{X}^{\mathrm{sm}}$ of the resulting map

$$\widehat{\mathcal{I}}\{-1\} \to (\zeta_p - 1)\cdot H^1(\widetilde{\Omega}_{\mathfrak{X}}) \tag{2.25.3}$$

extends uniquely over the entire $\mathfrak{X}$ to a map

$$\widehat{\Omega}^1_{\mathcal{X},\log}\{-1\} \to (\zeta_p - 1)\cdot H^1(\widetilde{\Omega}_{\mathfrak{X}}) \tag{2.25.4}$$

that is an isomorphism (equivalently, an epimorphism). The desired (2.22.1) will then follow.

Since the target of (2.25.4) has no nonzero local sections that vanish on $\mathfrak{X}^{\mathrm{sm}}$, the uniqueness is clear, so we may focus on the existence and may work locally on $\mathfrak{X}$. Moreover, the formation of the map (2.25.3) and of the source and the target of (2.25.4) commutes with étale localization on $\mathcal{X}$, so we may use (□) to assume that

$$\mathcal{X} = \operatorname{Spec}(\mathcal{O}[t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}]/(t_0 \cdots t_r - \pi)) \qquad \text{for some} \quad 0 \leqslant r \leqslant d. \tag{2.25.5}$$

Such an $\mathcal{X}$ may be compactified to a proper semistable $\mathcal{O}$-scheme, so, finally, we loose no generality by dropping (2.25.5) and assuming instead for the rest of the proof that $\mathcal{X}$ is proper.

In the proper case, by Theorem 2.23 and Remark 2.24, the map (2.25.3) algebraizes to

$$\mathcal{I}_{\mathcal{O}_C}\{-1\} \xrightarrow{f} \mathcal{F}$$

for some vector bundle $\mathcal{F}$ on $\mathcal{X}_{\mathcal{O}_C}$. By the Nakayama lemma and §2.20, f is surjective at every point of $\mathcal{X}^{\mathrm{sm}}_{\overline{k}}$.

Claim 2.25.6. There is an isomorphism $\mathcal{F}[\frac{1}{p}] \simeq \Omega^1_{\mathcal{X}_C/C}$.

Proof. By the adic GAGA [Sch13, 9.1 (i)], it suffices to find an analogous isomorphism after pullback to X^{ad}. Such a pullback of $\mathcal{F}[\frac{1}{p}]$ is isomorphic to $(R^1\nu_*\widehat{\mathcal{O}}^+_X)[\frac{1}{p}]$, and [Sch13, 6.19] supplies an isomorphism between $(R^1\nu_*\widehat{\mathcal{O}}^+_X)[\frac{1}{p}]$ and the pullback of $\Omega^1_{\mathcal{X}_C/C}$ to X^{ad}. □

Since $\mathcal{I}_{\mathcal{O}_C}\{-1\}[\frac{1}{p}] \simeq \Omega^1_{\mathcal{X}_C/C}$, too, Claim 2.25.6 ensures that $f[\frac{1}{p}]$ is a generically surjective morphism between isomorphic vector bundles on $\mathcal{X}_C$. Since $\mathcal{X}_C$ is proper and smooth, every global section of the structure sheaf of each connected component of $\mathcal{X}_C$ is constant, so $\det(f[\frac{1}{p}])$ is an isomorphism, to the effect that $f[\frac{1}{p}]$ is surjective on the entire $\mathcal{X}_C$.

In conclusion, $f|_{\mathcal{X}^{\mathrm{sm}}_{\mathcal{O}_C}}$ is a surjection between vector bundles of the same rank, so $f|_{\mathcal{X}^{\mathrm{sm}}_{\mathcal{O}_C}}$ is an isomorphism $\Omega^1_{\mathcal{X}^{\mathrm{sm}}_{\mathcal{O}_C}/\mathcal{O}_C}\{-1\} \cong \mathcal{F}|_{\mathcal{X}^{\mathrm{sm}}}$. Since $\mathcal{X}$ is Cohen–Macaulay and $\mathcal{X}\backslash\mathcal{X}^{\mathrm{sm}}$ is of codimension $\geqslant 2$ in $\mathcal{X}$, limit arguments and [EGA IV$_2$, 5.10.5] ensure that $\mathcal{F}$ is the unique vector bundle extension of $\mathcal{F}|_{\mathcal{X}^{\mathrm{sm}}_{\mathcal{O}_C}}$ to $\mathcal{X}_{\mathcal{O}_C}$. In particular, $f|_{\mathcal{X}^{\mathrm{sm}}_{\mathcal{O}_C}}$ extends to an isomorphism $(\Omega^1_{\mathcal{X},\log})_{\mathcal{O}_C}\{-1\} \cong \mathcal{F}$ whose formal p-adic completion gives the desired extension (2.25.4). □

Remark 2.26. By the proof above, we may (and do) choose the isomorphism (2.22.1) in such a way that its restriction to $\mathfrak{X}^{\mathrm{sm}}$ is compatible with the analogous isomorphism constructed in [BMS16, 8.3] in the smooth case. This compatibility determines the isomorphism (2.22.1) uniquely.

3. The object $A\Omega_{\mathfrak{X}}$ and its de Rham specialization

The goal of this section is to construct a certain object $A\Omega_{\mathfrak{X}}$ of the derived category of A_{inf}-module sheaves on $\mathfrak{X}_{\text{ét}}$ (see §3.2) and to identify its "de Rham specialization" with the formal p-adic completion of the logarithmic de Rham complex (see Theorem 3.15). We will use $A\Omega_{\mathfrak{X}}$ in §4 to define the A_{inf}-cohomology of $\mathcal{X}$.

Similarly to §2, in comparison to the smooth case treated in [BMS16], a major complication is the loss of flatness of the intermediate layers of the explicit perfectoid covers that we consider, which makes it difficult to apply the arguments of [BMS16, §9] directly. In fact, our approach is closer to that of [Bha16], the key step being the identification of the "Hodge–Tate specialization" of $A\Omega_{\mathfrak{X}}$ in Theorem 3.4. The main difference from op. cit. is in the way we transfer conclusions across (▫): in the smooth case such transfer is facilitated by a lemma of Kedlaya [Bha16, 4.9], which constructs *finite* étale analogues of (▫), whereas we base this transfer on Lemmas 3.10 and 3.11 below.

We begin by reviewing the basic notation that concerns the ring A_{inf} of integral p-adic Hodge theory.

3.1. The ring A_{inf}. We let $\mathcal{O}_C^\flat := \varprojlim_{y\mapsto y^p} \mathcal{O}_C/p$ be the tilt of $\mathcal{O}_C$ and set

$$A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat).$$

Reduction modulo p induces an isomorphism

$$\varprojlim_{y\mapsto y^p} \mathcal{O}_C \xrightarrow{\sim} \varprojlim_{y\mapsto y^p} \mathcal{O}_C/p = \mathcal{O}_C^\flat \tag{3.1.1}$$

of multiplicative monoids (cf. [Sch12, 3.4 (i)]), and for an $x \in \mathcal{O}_C^\flat$, we let $(\dots, x^{(1)}, x^{(0)})$ denote its preimage in $\varprojlim_{y \mapsto y^p} \mathcal{O}_C$. The map $x \mapsto \mathrm{val}_{\mathcal{O}_C}(x^{(0)})$ makes $\mathcal{O}_C^\flat$ a complete valuation ring of height 1 whose fraction field $C^\flat := \mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed (cf. [Sch12, 3.4 (iii), 3.7 (ii)]). The map

$$[x] \mapsto x^{(0)} \qquad \text{extends uniquely to a ring homomorphism} \qquad \theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C,$$

the *de Rham specialization* map, which is surjective. The *Hodge–Tate specialization* map is

$$\widetilde{\theta} := \theta \circ \varphi^{-1}\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C, \qquad \text{where} \quad \varphi\colon A_{\mathrm{inf}} \xrightarrow{\sim} A_{\mathrm{inf}} \quad \text{denotes the Witt vector Frobenius.}$$

If $\epsilon = (\dots, \zeta_{p^2}, \zeta_p, 1)$ is a compatible system of p-power roots of unity in $\mathcal{O}_C$, so that $\epsilon \in \mathcal{O}_C^\flat$ via (3.1.1), then $\xi := \sum_{i=0}^{p-1} [\epsilon^{i/p}]$ generates $\mathrm{Ker}\,\theta$ and $\widetilde{\xi} := \sum_{i=0}^{p-1} [\epsilon^i]$ generates $\mathrm{Ker}\,\widetilde{\theta}$ (cf. [BMS16, 3.16]).

We equip the local domain A_{inf} with the product of the valuation topologies via the Witt coordinate bijection $W(\mathcal{O}_C^\flat) \cong \prod_{n=1}^{\infty} \mathcal{O}_C^\flat$. Then A_{inf} is complete and its topology agrees with the $(p, [x])$-adic topology for any nonzero nonunit $x \in \mathcal{O}_C^\flat$. We will often use the element

$$\mu := [\epsilon] - 1 \in A_{\mathrm{inf}}, \qquad \text{for which the ideal} \qquad (\mu) \subset A_{\mathrm{inf}}$$

does not depend on the choice of ϵ (cf. [BMS16, 3.23]). Since $(p, \mu) = (p, [\epsilon - 1])$, the topology on A_{inf} is (p, μ)-adic. Since μ is not a zero divisor in $W_n(\mathcal{O}_C^\flat)$ for any $n \geqslant 1$, an inductive argument shows that every element of $\bigcap_{n \geqslant 1} (\mu, p^n) A_{\mathrm{inf}}$ lies in μA_{inf}, so A_{inf}/μ is p-adically complete by [SP, 031A].

With the notational setup of §3.1 in place, we are ready to introduce the protagonist of §3.

3.2. The object $A\Omega_{\mathfrak{X}}$. Letting $\mathbb{A}_{\mathrm{inf},\, X}$ on $X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}$ be a period sheaf of [Sch13, 6.1 (i)], we set

$$A\Omega_{\mathfrak{X}} := L\eta_{(\mu)}(R\nu_* \mathbb{A}_{\mathrm{inf},\, X}) \in D^{\geqslant 0}(\mathfrak{X}_{\mathrm{\acute{e}t}}, A_{\mathrm{inf}}),$$

where the décalage functor $L\eta$ of [BMS16, §6] is taken with respect to the ideal (μ) of the constant sheaf A_{inf} on $\mathfrak{X}_{\mathrm{\acute{e}t}}$. Since $\varphi(\eta) = \widetilde{\xi}\eta$, the Frobenius automorphism of $\mathbb{A}_{\mathrm{inf},\, X}$ gives the isomorphism

$$(A\Omega_{\mathfrak{X}} \otimes_{A_{\mathrm{inf}},\, \varphi} A_{\mathrm{inf}})[\tfrac{1}{\widetilde{\xi}}] \cong \left(L\eta_{(\varphi(\eta))}(R\nu_* \mathbb{A}_{\mathrm{inf},\, X})\right)[\tfrac{1}{\widetilde{\xi}}] \overset{\text{[BMS16, 6.11]}}{\cong} (A\Omega_{\mathfrak{X}})[\tfrac{1}{\widetilde{\xi}}]. \tag{3.2.1}$$

To prepare for the analysis of the de Rham specialization of $A\Omega_{\mathfrak{X}}$ in Theorem 3.15, we first relate the Hodge–Tate specialization to the object $\widetilde{\Omega}_{\mathfrak{X}}$ studied in §2.

3.3. The Hodge–Tate specialization map. The sheaf $\mathbb{A}_{\mathrm{inf},\, X}$ comes equipped with the map $\theta_X\colon \mathbb{A}_{\mathrm{inf},\, X} \to \widehat{\mathcal{O}}_X^+$ that is compatible with the map $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C$ discussed in §3.1. By [Sch13, 6.3, 6.5, and 4.7], θ_X is surjective and $\mathrm{Ker}(\theta_X) = \xi \cdot \mathbb{A}_{\mathrm{inf},\, X}$. Therefore, continuing to denote the Frobenius by φ, we have that the map

$$\widetilde{\theta}_X := \theta_X \circ \varphi^{-1}\colon \mathbb{A}_{\mathrm{inf},\, X} \twoheadrightarrow \widehat{\mathcal{O}}_X^+ \qquad \text{satisfies} \qquad \mathrm{Ker}(\widetilde{\theta}_X) = \widetilde{\xi} \cdot \mathbb{A}_{\mathrm{inf},\, X}.$$

The projection formula [SP, 0944] then provides the identification

$$R\nu_* \mathbb{A}_{\mathrm{inf},\, X} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \widetilde{\theta}} \mathcal{O}_C \cong R\nu_* \widehat{\mathcal{O}}_X^+. \tag{3.3.1}$$

Since $\widetilde{\theta}(\mu) = \zeta_p - 1$ is a nonzero divisor in $\mathcal{O}_C$, we obtain the specialization map

$$A\Omega_{\mathfrak{X}} = L\eta_{(\mu)}(R\nu_* \mathbb{A}_{\mathrm{inf},\, X}) \to L\eta_{(\zeta_p - 1)}(R\nu_* \mathbb{A}_{\mathrm{inf},\, X} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \widetilde{\theta}} \mathcal{O}_C) \cong \widetilde{\Omega}_{\mathfrak{X}} \tag{3.3.2}$$

as in [Bha16, 5.14], where we use (3.3.1) and [BMS16, 6.14] for the last identification.

Theorem 3.4. *The map (3.3.2) induces an isomorphism* $A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \widetilde{\theta}} \mathcal{O}_C \xrightarrow{\sim} \widetilde{\Omega}_{\mathfrak{X}}$.

We will indirectly deduce the claim from the criterion supplied by [Bha16, 5.14], so the key point will be to prove that suitable variants of the complex $A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ have p-torsion free cohomology. We will accomplish this in Lemma 3.11 (b) after the following preparations.

The claim of Theorem 3.4 is étale local on $\mathfrak{X}$, so we may assume that $\mathfrak{X} = \mathrm{Spf}\, R$, that $\mathfrak{X}$ is connected, and that there is an étale $\mathcal{O}_C$-morphism

$$\mathfrak{X} = \mathrm{Spf}\, R \to \mathrm{Spf}\, R^{\square} \qquad \text{with} \qquad R^{\square} = \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - \pi) \tag{3.4.1}$$

for some $0 \leqslant r \leqslant d$ and a uniformizer $\pi \in \mathcal{O}$. We impose these assumptions until Theorem 3.15 (i.e., until the end of the proof of Theorem 3.4) and we begin by examining the sheaf $\mathbb{A}_{\mathrm{inf},X}$ on a suitable perfectoid cover of R.

3.5. The tilt $R^{\flat}_{\infty}$. We fix a system $\pi^{\flat} := (\dots, \pi^{1/p}, \pi)$ of compatible p-power roots of π in $\mathcal{O}_C$ and view $\pi^{\flat}$ as an element of $\mathcal{O}_C^{\flat}$. As in §2.3, for an $m \in \mathbb{Z}_{\geqslant 0}$ we set

$$R^{\square}_m := \mathcal{O}_C\{t_0^{1/p^m}, \dots, t_r^{1/p^m}, t_{r+1}^{\pm 1/p^m}, \dots, t_d^{\pm 1/p^m}\}/(t_0^{1/p^m} \cdots t_r^{1/p^m} - \pi^{1/p^m}) \quad \text{with} \quad R^{\square}_{\infty} := \widehat{\varinjlim R^{\square}_m}$$

and, similarly,

$$R_m := R \otimes_{R^{\square}} R^{\square}_m \qquad \text{with} \qquad R_{\infty} := \widehat{\varinjlim R_m} \cong (R \otimes_{R^{\square}} R^{\square}_{\infty})\widehat{\ }.$$

Similarly to [Sch12, proof of Prop. 5.20], the tilt $(R^{\square}_{\infty})^{\flat}$ of the perfectoid ring $R^{\square}_{\infty}$ is given by

$$\begin{aligned}(R^{\square}_{\infty})^{\flat} &= \left(\varinjlim_m \mathcal{O}_C^{\flat}[x_0^{1/p^m}, \dots, x_r^{1/p^m}, x_{r+1}^{\pm 1/p^m}, \dots, x_d^{\pm 1/p^m}]/(x_0^{1/p^m} \cdots x_r^{1/p^m} - (\pi^{\flat})^{1/p^m})\right)\widehat{\ } \\ &\cong \widehat{\bigoplus}_{\substack{(a_0, \dots, a_d) \in \mathbb{Z}[\frac{1}{p}]_{\geqslant 0}^{\oplus (r+1)} \oplus \mathbb{Z}[\frac{1}{p}]^{\oplus (d-r)}, \\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} \mathcal{O}_C^{\flat} \cdot x_0^{a_0} \cdots x_d^{a_d}\end{aligned} \tag{3.5.1}$$

where the completions are $\pi^{\flat}$-adic and the decomposition is as $\mathcal{O}_C^{\flat}$-modules. Likewise,

$$\text{the tilt} \quad R^{\flat}_{\infty} \quad \text{of the perfectoid ring} \quad R_{\infty}$$

identifies with the $\pi^{\flat}$-adic completion of any lift of the étale $R^{\square}_{\infty}/\pi$-algebra R_{∞}/π to an étale $(R^{\square}_{\infty})^{\flat}$-algebra (such a lift exists, see [SP, 04D1]).

3.6. The ring $\mathbb{A}_{\mathrm{inf}}(R_{\infty})$. By [Bha16, 2.5], for a perfect $\mathbb{F}_p$-algebra A, the Witt ring $W(A)$ is the unique p-adically complete p-torsion free $\mathbb{Z}_p$-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/p \simeq A$. By [Bha16, 2.4], for an $a \in A$, the Teichmüller $[a] \in \widetilde{A}$ equals $\lim_{n \to \infty} \widetilde{a}_n^{p^n}$ where $\widetilde{a}_n \in \widetilde{A}$ is any lift of a^{1/p^n}.

Therefore, the period ring

$$\mathbb{A}_{\mathrm{inf}}(R^{\square}_{\infty}) := W((R^{\square}_{\infty})^{\flat})$$

is given by

$$\begin{aligned}\mathbb{A}_{\mathrm{inf}}(R^{\square}_{\infty}) &= \left(\varinjlim_m A_{\mathrm{inf}}[X_0^{1/p^m}, \dots, X_r^{1/p^m}, X_{r+1}^{\pm 1/p^m}, \dots, X_d^{\pm 1/p^m}]/(X_0^{1/p^m} \cdots X_r^{1/p^m} - [(\pi^{\flat})^{1/p^m}])\right)\widehat{\ } \\ &\cong \widehat{\bigoplus}_{\substack{(a_0, \dots, a_d) \in \mathbb{Z}[\frac{1}{p}]_{\geqslant 0}^{\oplus (r+1)} \oplus \mathbb{Z}[\frac{1}{p}]^{\oplus (d-r)}, \\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d}\end{aligned}$$

where the completions are (p, μ)-adic and the decomposition is as A_{inf}-modules; in terms of (3.5.1), each X_i^{1/p^m} identifies with the Teichmüller $[x_i^{1/p^m}]$.

Lemma 3.6.1. *The period ring*

$$\mathbb{A}_{\mathrm{inf}}(R_\infty) := W(R_\infty^\flat)$$

identifies with the (p,μ)*-adic completion of any lift of the étale* $R_\infty^\square/\pi$*-algebra* R_∞/π *to an étale* $W((R_\infty^\square)^\flat)$*-algebra. Moreover,* $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$ *is* p*-adically complete.*

Proof. Similarly to the case of A_{inf} discussed in §3.1, we use the Witt coordinate bijection

$$W(R_\infty^\flat) \cong \textstyle\prod_{n=1}^\infty R_\infty^\flat$$

and the $\pi^\flat$-adic topology on $R_\infty^\flat$ to topologize $W(R_\infty^\flat)$, and we see that this topology agrees with the (p,μ)-adic topology and that $W(R_\infty^\flat)$ is (p,μ)-adically complete. Moreover, by §3.5, $W(R_\infty^\flat)/I$ is $W((R_\infty^\square)^\flat)/I$-étale whenever $I = (p, \mu^n)$ with $n \geqslant 1$. Thus, since each $W_m(R_\infty^\flat)$ is μ-torsion free, reduction modulo μ^n does not interfere with the p-adic filtration of $W_m(R_\infty^\flat)$, to the effect that [BouAC, Ch. III, §5.2, Thm. 1 (i)⇔(iv)] implies the étaleness whenever $I = (p^m, \mu^n)$ with $m, n \geqslant 1$, and hence also whenever $I = (p,\mu)^n$ with $n \geqslant 1$. Since $W(R_\infty^\flat)$ is (p,μ)-adically complete, the first claim follows. As in §3.1, the p-adic completeness of $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$ follows from the μ-torsion freeness of all the $W_m(R_\infty^\flat)$. ☐

3.7. The action of Δ. As in §2.4, the group

$$\Delta := \left\{ (\epsilon_0, \dots, \epsilon_d) \in \left(\varprojlim_{m \geqslant 0} \mu_{p^m}(\mathcal{O}_C) \right)^{\oplus (d+1)} \,\middle|\, \epsilon_0 \cdots \epsilon_r = 1 \right\} \simeq \mathbb{Z}_p^{\oplus d}$$

acts $R^\square$-equivariantly on $R_\infty^\square$, and hence also compatibly and R-equivariantly on R_∞. The induced continuous Δ-action on $(R_\infty^\square)^\flat$ preserves the decomposition

$$(R_\infty^\square)^\flat \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d) \in \mathbb{Z}[\frac{1}{p}]_{\geqslant 0}^{\oplus(r+1)} \oplus \mathbb{Z}[\frac{1}{p}]^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} \mathcal{O}_C^\flat \cdot x_0^{a_0} \cdots x_d^{a_d} \qquad (3.7.1)$$

and an $(\epsilon_0, \dots, \epsilon_d) \in \Delta$ acts on $x_j^{a_j}$ by scaling by $\epsilon_j^{a_j} \in \mathcal{O}_C^\flat$. Therefore, the induced continuous Δ-action on $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$ respects the decomposition

$$\mathbb{A}_{\mathrm{inf}}(R_\infty^\square) \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d) \in \mathbb{Z}[\frac{1}{p}]_{\geqslant 0}^{\oplus(r+1)} \oplus \mathbb{Z}[\frac{1}{p}]^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d} \qquad (3.7.2)$$

and an $(\epsilon_0, \dots, \epsilon_d) \in \Delta$ acts on $X_j^{a_j}$ by scaling by $[\epsilon_j^{a_j}] \in A_{\mathrm{inf}}$.

3.8. The comparison to continuous group cohomology. As in §2.4,

$$X_\infty^\square = \text{“}\varprojlim \mathrm{Spa}(R_m^\square[\tfrac{1}{p}], R_m^\square)\text{”} \qquad \text{and} \qquad X_\infty^{\mathrm{ad}} = \text{“}\varprojlim \mathrm{Spa}(R_m[\tfrac{1}{p}], R_m)\text{”}$$

are affinoid perfectoid pro-(finite étale Galois) Δ-covers of the adic generic fibers $X^\square$ and X^{ad} of $\mathrm{Spf}\, R^\square$ and $\mathrm{Spf}\, R$, respectively. By [Sch13, 3.5, 3.7 (iii) and its proof, 6.5 (i), 6.6], the Čech complex with coefficients in $\mathbb{A}_{\mathrm{inf},X}$ (resp., in $\mathbb{A}_{\mathrm{inf},X^\square}$) of the cover $X_\infty^{\mathrm{ad}} \to X^{\mathrm{ad}}$ (resp., $X_\infty^\square \to X^\square$) identifies with the continuous cochain complex $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))$ (resp., $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty^\square))$). We consider the resulting edge morphism (cf. [SP, 08BN])

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \xrightarrow{e} R\Gamma(X^{\mathrm{ad}}_{\mathrm{proét}}, \mathbb{A}_{\mathrm{inf},X}) \quad (\text{resp.,} \quad R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty^\square)) \xrightarrow{e^\square} R\Gamma(X^\square_{\mathrm{proét}}, \mathbb{A}_{\mathrm{inf},X^\square})).$$

By the almost purity theorem, i.e., by [Sch13, 6.5 (ii)], the ideal $[\mathfrak{m}^\flat]A_{\mathrm{inf}} \subset A_{\mathrm{inf}}$ generated by the Teichmüllers of the elements of the maximal ideal $\mathfrak{m}^\flat \subset \mathcal{O}_C^\flat$ kills $H^i(\mathrm{Cone}(e))$ and $H^i(\mathrm{Cone}(e^\square))$ for $i \in \mathbb{Z}$. Since $\mu \in W(\mathfrak{m}^\flat)$ with $W(\mathfrak{m}^\flat) := \mathrm{Ker}(W(\mathcal{O}_C^\flat) \twoheadrightarrow W(\overline{k}))$ and $A\Omega_{\mathfrak{X}}$ is built using $L\eta_{(\mu)}$, it will be useful to upgrade this annihilation result as follows.

Lemma 3.8.1. *For* $i \in \mathbb{Z}$*, the ideal* $W(\mathfrak{m}^\flat) \subset A_{\mathrm{inf}}$ *kills* $H^i(\mathrm{Cone}(e))$ *and* $H^i(\mathrm{Cone}(e^\square))$.

Proof. We argue similarly to [BMS16, proof of Thm. 5.6]. By [Sch13, 3.7 (iii)], we may view $R\Gamma_{\mathrm{cont}}(\Delta, -)$ as the derived global sections functor of the site of profinite Δ-sets. Therefore, since $\mathbb{A}_{\mathrm{inf},X}$ is p-adically complete and takes p-adically complete values on affinoid perfectoids (cf. [Sch13, 4.7 and 6.5 (i)]), both $R\Gamma(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X})$ and $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))$ are derived p-adically complete (cf. §1.2), and hence, by [BS15, 3.4.4 and 3.4.14], so is each $H := H^i(\mathrm{Cone}(e))$, and likewise for $e^\square$. Then any free resolution $F^\bullet$ of the A_{inf}-module H placed in degree 0 satisfies

$$\mathrm{Coker}(\varprojlim_n F^{-1}/p^n \to \varprojlim_n F^0/p^n) \cong H.$$

The $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$-annihilation of H implies that for every $n \geqslant 1$ the ideal $[\mathfrak{m}^\flat]W_n(\mathcal{O}_C^\flat) = W_n(\mathfrak{m}^\flat)$ of $W_n(\mathcal{O}_C^\flat)$ kills both

$$H/p^n \cong \mathrm{Coker}(F^{-1}/p^n \to F^0/p^n) \cong H^0(F^\bullet \otimes \mathbb{Z}/p^n\mathbb{Z}) \qquad \text{and} \qquad H^{-1}(F^\bullet \otimes \mathbb{Z}/p^n\mathbb{Z}).$$

Thus, since $W_n(\mathfrak{m}^\flat)^2 = W_n(\mathfrak{m}^\flat)$, an inductive argument shows that $W(\mathfrak{m}^\flat) \cdot \varprojlim_n F^0/p^n$ lies in the image of $\varprojlim_n F^{-1}/p^n$, i.e., that $W(\mathfrak{m}^\flat)$ kills H, as desired. ☐

We will use Lemma 3.8.1 through the following variant of [Bha16, 6.14].

Lemma 3.9. *If $A \xrightarrow{a} B$ is a morphism in $D(A_{\mathrm{inf}})$ such that $H^i(\mathrm{Cone}(a))$ is killed by $W(\mathfrak{m}^\flat)$ and $H^i(A \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu)[W(\mathfrak{m}^\flat)] = 0$ for all i, then the map $L\eta_{(\mu)}(a)$ is an isomorphism.*

Proof. Since $L\eta$ is not in general a triangulated functor, the fact that $L\eta_{(\mu)}(\mathrm{Cone}(a)) \cong 0$ does not a priori suffice. Nevertheless, the argument used to prove [Bha16, 6.14] gives the claim. ☐

The following two lemmas investigate the properties of the cohomology groups $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu)$ and $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$, respectively, and form the heart of the proof of Theorem 3.4.

Lemma 3.10.

(a) *Let $A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d}$ be a summand in (3.7.2) and let $m \in \mathbb{Z}_{\geqslant 0}$ be minimal such that $p^m(a_0 - a_j) \in \mathbb{Z}$ for $1 \leqslant j \leqslant r$ and $p^m a_j \in \mathbb{Z}$ for $r+1 \leqslant j \leqslant d$. For $i \in \mathbb{Z}$, we have*

$$H^i_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d}) \simeq \textstyle\bigoplus_I A_{\mathrm{inf}}/\varphi^{-m}(\mu) \quad \text{as } A_{\mathrm{inf}}\text{-modules for some set } I. \tag{3.10.1}$$

In particular, $H^i_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d})$ is p-torsion free.

(b) *For $i \in \mathbb{Z}$, the group $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu)$ is p-torsion free and for every $n \in \mathbb{Z}_{\geqslant 1}$ the map*

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu) \otimes_{A_{\mathrm{inf}}/\mu} A_{\mathrm{inf}}/(\mu, p^n) \to H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/(\mu, p^n)) \tag{3.10.2}$$

is an isomorphism. Moreover, there exist set J and integers $n_j \in \mathbb{Z}_{\geqslant 0}$ for $j \in J$ such that

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/(\mu, p^n)) \simeq \textstyle\bigoplus_{j \in J} A_{\mathrm{inf}}/(\varphi^{-n_j}(\mu), p^n) \quad \text{as } A_{\mathrm{inf}}\text{-modules.} \tag{3.10.3}$$

Proof.

(a) The following argument builds on the ideas of [Bha16, proof of Lem. 4.6]. The group Δ is topologically generated by the following d elements (see §3.1 for the definition of ϵ):

$(\epsilon, 1, \ldots, 1, \epsilon^{-1}, 1, \ldots, 1)$ with 0^{th} and j^{th} nonidentity entries, where $j = 1, \ldots, r$;

$(1, \ldots, 1, \epsilon, 1, \ldots, 1)$ with j^{th} nonidentity entry, where $j = r+1, \ldots, d$.

We set

$$b_j := a_0 - a_j \quad \text{for} \quad 1 \leqslant j \leqslant r \qquad \text{and} \qquad b_j := a_j \quad \text{for} \quad r+1 \leqslant j \leqslant d,$$

so that $m \in \mathbb{Z}_{\geqslant 0}$ is minimal such that $p^m b_j \in \mathbb{Z}$ for all j. Then, by Lemma 2.7, for the A_{inf}/μ-tensor product $C^\bullet$ of the d complexes

$$[A_{\mathrm{inf}}/\mu \xrightarrow{[\epsilon^{b_j}]-1} A_{\mathrm{inf}}/\mu] \cong A_{\mathrm{inf}}/([\epsilon^{b_j}]-1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu, \tag{3.10.4}$$

there is an A_{inf}-module isomorphism $H^i(C^\bullet) \simeq H^i_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d})$.

If $m = 0$, then $b_j \in \mathbb{Z}$, and hence also $\mu \mid [\epsilon^{b_j}] - 1$, for all j. Thus, in this case $C^\bullet$ is a direct sum of shifts of A_{inf}/μ (with vanishing differentials) and the claim follows.

If $m > 0$, then we may assume that for all j we have $b_j/b_1 \in \mathbb{Z}_{(p)}$, so that $b_1 \notin \mathbb{Z}$ and both $[\epsilon^{b_1}] - 1 \mid [\epsilon^{b_j}] - 1$ and $[\epsilon^{b_1}] - 1 \mid \mu$. By resolving A_{inf}/μ in (3.10.4) with $j = 1$, we then see that $C^\bullet$ is quasi-isomorphic to a direct sum of shifts of $A_{\mathrm{inf}}/([\epsilon^{b_1}] - 1) \cong A_{\mathrm{inf}}/\varphi^{-m}(\mu)$.

(b) Since p, μ is a regular sequence in A_{inf}, (3.7.2) gives the Δ-decomposition

$$\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d)\in \mathbb{Z}[\frac{1}{p}]^{\oplus(r+1)}_{\geqslant 0} \oplus \mathbb{Z}[\frac{1}{p}]^{\oplus(d-r)},\\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d} \tag{3.10.5}$$

in which the completion is p-adic. Lemma 2.6 (i) then combines with (a) to prove that $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu)$ is p-torsion free.

By Lemma 2.7, $H^*_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu)$ is computed by a Koszul complex $D^\bullet$ whose terms are finite direct sums of copies of $\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu$, whereas $H^*_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/(\mu, p^n))$ is computed by its $A_{\mathrm{inf}}/(\mu, p^n)$-base change $D^\bullet/p^n$. In fact, since both A_{inf}/μ and the terms of $D^\bullet$ are p-torsion free, these terms are acyclic for $- \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}/\mu} A_{\mathrm{inf}}/(\mu, p^n)$, so that, by the second spectral sequence in [SP, 061Z], the levelwise base change $D^\bullet/p^n$ identifies with $D^\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}/\mu} A_{\mathrm{inf}}/(\mu, p^n)$. Therefore, by the first spectral sequence in [SP, 061Z], the p-torsion freeness of $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu)$ implies that (3.10.2) is an isomorphism.

An analogous (but simpler) argument also proves the base change (3.10.2) for the groups $H^i_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d})$ with $A_{\mathrm{inf}}/\mu \cdot X_0^{a_0} \cdots X_d^{a_d}$ as in (a). Therefore, (3.10.3) follows from the modulo p^n reduction of the decomposition (3.10.5) and from (3.10.1). ☐

Lemma 3.11.

(a) For $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geqslant 1}$, the map

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n)) \to H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^{n-1})) \tag{3.11.1}$$

is surjective and its kernel is the p-torsion subgroup of $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n))$. In particular, $R^1 \varprojlim_n H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n)) = 0$ for every $i \in \mathbb{Z}$.

(b) For $i \in \mathbb{Z}$, the A_{inf}-module $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$ has no nonzero p-torsion or $W(\mathfrak{m}^\flat)$-torsion.

Proof.

(a) The group Δ acts trivially on the subring

$$A^\square_n := (A_{\mathrm{inf}}/(\mu, p^n))[X_0, \dots, X_r, X^{\pm 1}_{r+1}, \dots, X^{\pm 1}_d]/(X_0 \cdots X_r - [\pi^\flat]) \subset \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/(\mu, p^n).$$

Therefore, if we let A_n be the étale $A^\square_n$-algebra that lifts a reduction of the étale $R^\square/p$-algebra R/p, then Lemma 3.6.1 supplies a Δ-equivariant identification

$$\mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n) \cong \mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/(\mu, p^n) \otimes_{A^\square_n} A_n.$$

Lemma 2.7 then gives compatible identifications

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R^{\square}_{\infty})/(\mu, p^n)) \otimes_{A^{\square}_n} A_n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p^n)) \qquad \text{for } n \geqslant 1,$$

which reduce us to the case $R = R^{\square}$. In this case, Lemma 3.10 (b) supplies the conclusion.

(b) By Lemma 3.6.1, $\mathbb{A}_{\mathrm{inf}}(R_{\infty})/\mu$ is p-adically complete, so $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/\mu)$ is derived p-adically complete (cf. §1.2 and the proof of Lemma 3.8.1). Thus, by (a) and [SP, 08U5, 0944],

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/\mu) \xrightarrow{\sim} \varprojlim_n H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p^n)) \quad \text{for every } i \in \mathbb{Z}, \tag{3.11.2}$$

so the claimed p-torsion freeness follows from (a).

Since μ is not a zero divisor in $W_n(R^{\flat}_{\infty})$ for $n \geqslant 1$, the sequences

$$0 \to \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p) \xrightarrow{p^n} \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p^{n+1}) \to \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p^n) \to 0$$

are exact. By (a), $H^i_{\mathrm{cont}}(\Delta, -)$ preserves their exactness, so $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p^n))$ is a successive extension of copies of $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/(\mu, p))$. Thus, due to (3.11.2), it remains to prove that $H^i_{\mathrm{cont}}(\Delta, R^{\flat}_{\infty}/\mu)$ has no nonzero $\mathfrak{m}^{\flat}$-torsion. In fact, we will prove the same for $H^i_{\mathrm{cont}}(\Delta, R^{\flat}_{\infty}/a)$ with any $a \in \mathfrak{m}^{\flat}$, so the Frobenius allows us to assume that $\pi^{\flat} \mid a$ in $\mathcal{O}^{\flat}_C$.

Similarly to the proof of (a), since $R^{\flat}_{\infty}/\pi^{\flat} \cong R^{\square}_{\infty}/\pi \otimes_{R^{\square}/\pi} R/\pi$, the $\pi^{\flat} \mid a$ assumption gives

$$H^i_{\mathrm{cont}}(\Delta, R^{\flat}_{\infty}/a) \cong H^i_{\mathrm{cont}}(\Delta, R^{\square}_{\infty}/a^{\sharp}) \otimes_{R^{\square}/\pi} R/\pi \quad \text{for some} \quad a^{\sharp} \in \mathcal{O}_C \quad \text{with} \quad \pi \mid a^{\sharp}.$$

To deduce the sought $\mathfrak{m}$-torsion freeness of $H^i_{\mathrm{cont}}(\Delta, R^{\flat}_{\infty}/a)$, analogously to the proof of Lemma 2.12, one decomposes $H^i_{\mathrm{cont}}(\Delta, R^{\square}_{\infty}/a^{\sharp})$ into a direct sum of $R^{\square}/\pi$-modules, each one of which may be defined over a discretely valued subring of $\mathcal{O}_C$, and one applies Lemma 2.9. ☐

The following proposition is a key consequence of our work in the preceding lemmas.

Proposition 3.12. *For the edge morphism e of §3.8, the map $L\eta_{(\mu)}(e)$ is an isomorphism:*

$$L\eta_{(\mu)}(e)\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty}))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma(X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},\, X})).$$

Proof. By the projection formula [SP, 0944],

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})/\mu), \tag{3.12.1}$$

so Lemma 3.11 (b) implies that the cohomology of $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ has no nonzero $W(\mathfrak{m}^{\flat})$-torsion. The claim then follows by combining Lemmas 3.8.1 and 3.9. ☐

It will be useful (especially in the proof of Proposition 4.6) to know that $A\Omega_{\mathfrak{X}}$ is derived $(\operatorname{Ker}\theta)$-adically complete. Such completeness is not automatic: $L\eta_{(\mu)}$ need not commute with derived limits; however, by [BMS16, 6.19], $L\eta$ preserves derived completeness when used in the context of a *replete* topos. To reduce to such a context, we will use a presheaf topos to relate $A\Omega_{\mathfrak{X}}$ to its presheaf analogue, similarly to [BMS16, proof of Prop. 9.14]. This relation will also be useful in the proof of Theorem 3.4.

3.13. The presheaf version $A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$. Continuing to assume that $\mathfrak{X}$ is as in (3.4.1), we consider the topos $\mathfrak{X}^{\mathrm{psh}}_{\mathrm{\acute{e}t}}$ of presheaves on the category of affine étale formal $\mathfrak{X}$-schemes. We let $\phi\colon \mathfrak{X}^{\mathrm{shv}}_{\mathrm{\acute{e}t}} \to \mathfrak{X}^{\mathrm{psh}}_{\mathrm{\acute{e}t}}$ be the morphism from the topos of sheaves, so that ϕ_* is forgetful and ϕ^* is the sheafification. We set

$$\nu^{\mathrm{psh}} := \phi \circ \nu\colon X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}^{\mathrm{psh}}_{\mathrm{\acute{e}t}} \qquad \text{and} \qquad A\Omega^{\mathrm{psh}}_{\mathfrak{X}} := L\eta_{(\mu)}(R\nu^{\mathrm{psh}}_{*}\mathbb{A}_{\mathrm{inf},\, X}),$$

so that, by [BMS16, 6.14], $\phi^*(A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \cong A\Omega_{\mathfrak{X}}$. Since $L\eta_{(\mu)}$ and $R\Gamma(\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}, -)$ commute,

$$R\Gamma(\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \cong L\eta_{(\mu)} R\Gamma(X_{\text{proét}}^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},X}),$$

and likewise for hypercohomology over other affine étale formal $\mathfrak{X}$-schemes.

Proposition 3.14. *Assume the setup of §3.13 and let $\widetilde{\Omega}_{\mathfrak{X}} \in D^{\geqslant 0}(\mathcal{O}_{\mathfrak{X},\text{ét}})$ be the object defined in §2.1.*

(a) There is an identification $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C \cong R\phi_* \widetilde{\Omega}_{\mathfrak{X}}$ in $D(\mathfrak{X}_{\text{ét}}^{\mathrm{psh}})$.

(b) The map $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \to R\phi_* A\Omega_{\mathfrak{X}}$ is an isomorphism and $A\Omega_{\mathfrak{X}}$ is derived $(\mathrm{Ker}\,\theta)$-adically complete.

Proof.

(a) By Lemma 3.11 (b) and (3.12.1), the cohomology of $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ is p-torsion free. Thus, since $\widetilde{\xi} \equiv p \bmod (\mu)$ (cf. §3.1), [Bha16, 5.14 and its proof] imply that

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C).$$

Corollary 2.13 and Proposition 3.12 then imply that

$$L\eta_{(\mu)}(R\Gamma(X_{\text{proét}}^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},X})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\Gamma(X_{\text{proét}}^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C),$$

and likewise for other affine étale formal $\mathfrak{X}$-schemes. Thus,

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\nu_*^{\mathrm{psh}}(\mathbb{A}_{\mathrm{inf},X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C) \cong L\eta_{(\zeta_p - 1)}(R\nu_*^{\mathrm{psh}}(\widehat{\mathcal{O}}_X^+)). \quad (3.14.1)$$

It remains to apply Remark 2.14 to conclude that the sheafification map

$$L\eta_{(\zeta_p - 1)}(R\nu_*^{\mathrm{psh}}(\widehat{\mathcal{O}}_X^+)) \to R\phi_*(\phi^*(L\eta_{(\zeta_p - 1)}(R\nu_*^{\mathrm{psh}}(\widehat{\mathcal{O}}_X^+)))) \overset{\text{[BMS16, 6.14]}}{\cong} R\phi_*(\widetilde{\Omega}_{\mathfrak{X}})$$

induces an isomorphism on all the cohomology groups, and hence is an isomorphism.

(b) By [SP, 090T], $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ is $\widetilde{\xi}$-adically complete. Therefore, [BMS16, 6.19] ensures that $L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)))$ is derived $\widetilde{\xi}$-adically complete, and likewise for other affine étale formal $\mathfrak{X}$-schemes. Proposition 3.12 then implies that $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ is derived $\widetilde{\xi}$-adically complete and, analogously, also derived ξ-adically complete.

By (a), the sheafification map

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C \to R\phi_*(\phi^*(A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C))$$

is an isomorphism. Thus, since $\mathrm{Ker}\,\widetilde{\theta} = (\widetilde{\xi})$ and $\widetilde{\xi}$ is not a zero divisor, the five lemma implies that the sheafification map

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\widetilde{\xi}^n) \to R\phi_*(\phi^*(A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\widetilde{\xi}^n)))$$

is an isomorphism for every $n \geqslant 1$. The derived $\widetilde{\xi}$-adic completeness of $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ and the $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$-analogue of [BMS16, 9.15] then ensure that

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \xrightarrow{\sim} R\phi_*(\phi^*(A\Omega_{\mathfrak{X}}^{\mathrm{psh}})) \cong R\phi_*(A\Omega_{\mathfrak{X}}). \quad (3.14.2)$$

To deduce the derived ξ-adic completeness of $A\Omega_{\mathfrak{X}}$ from that of $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$, we apply $R\Gamma(\mathfrak{X}'_{\text{ét}}, -)$ to (3.14.2) for a variable affine étale formal $\mathfrak{X}$-scheme $\mathfrak{X}'$. ∎

We proved a presheaf version of Theorem 3.4 in (3.14.1); this implies the sheaf version as follows.

Proof of Theorem 3.4. By [BMS16, 6.14], $L\eta_{(\zeta_p - 1)}$ commutes with ϕ^*. Therefore, the desired claim follows by applying ϕ^* to the isomorphism (3.14.1). ☐

With Theorem 3.4 in hand, we are ready to identify the de Rham specialization of $A\Omega_{\mathfrak{X}}$.

Theorem 3.15. *Endow $\mathcal{X}$ with the log structure that arises from the inclusion of the divisor $\mathcal{X}_k$ (cf. §2.21), and let $\Omega^\bullet_{\mathcal{X},\log}$ be the resulting logarithmic de Rham complex. Letting $\widehat{\Omega}^\bullet_{\mathcal{X},\log}$ denote the formal p-adic completion of the base change of $\Omega^\bullet_{\mathcal{X},\log}$ to $\mathcal{X}_{\mathcal{O}_C}$, we have*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong \widehat{\Omega}^\bullet_{\mathcal{X},\log}.$$

Proof. Similarly to [BMS16, proof of Thm. 14.1], since $\theta = \widetilde{\theta} \circ \varphi$ and $\varphi(\mu) = \widetilde{\xi}\mu$ (cf. §3.1), we have

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \overset{\text{[BMS16, 6.11]}}{\cong} (L\eta_{(\widetilde{\xi})}(A\Omega_{\mathfrak{X}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C. \quad (3.15.1)$$

By [BMS16, 6.12], $(L\eta_{(\widetilde{\xi})}(A\Omega_{\mathfrak{X}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C$ identifies with the complex whose i^{th} degree term is

$$H^i(A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C) \otimes_{\mathcal{O}_C} \left(\tfrac{\mathrm{Ker}\,\widetilde{\theta}}{(\mathrm{Ker}\,\widetilde{\theta})^2}\right)^{\otimes i} \overset{3.4}{\cong} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \otimes_{\mathcal{O}_C} \left(\tfrac{\mathrm{Ker}\,\widetilde{\theta}}{(\mathrm{Ker}\,\widetilde{\theta})^2}\right)^{\otimes i}$$

and the differentials are given by Bockstein homomorphisms.

The perfectness of $\mathcal{O}_C^\flat$ implies that $\widehat{\mathbb{L}}_{A_{\mathrm{inf}}/\mathbb{Z}_p} \cong 0$ and (2.18.1) (applied with $\mathfrak{X} = \mathrm{Spf}\,\mathcal{O}_C$) implies that $\widehat{\mathbb{L}}_{\mathcal{O}_C/\mathbb{Z}_p} \cong \mathcal{O}_C\{1\}[1]$, so $\widehat{\mathbb{L}}_{\mathcal{O}_C/A_{\mathrm{inf}}} \cong \mathcal{O}_C\{1\}[1]$ where $\mathcal{O}_C$ is regarded as an A_{inf}-algebra via $\widetilde{\theta}$. This combines with [Ill71, III.3.2.4 (iii)] to supply an isomorphism $\frac{\mathrm{Ker}\,\widetilde{\theta}}{(\mathrm{Ker}\,\widetilde{\theta})^2} \cong \mathcal{O}_C\{1\}$. In conclusion, due to Theorem 2.22 and the previous paragraph, $A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C$ identifies with the complex whose i^{th} degree term is $\widehat{\Omega}^i_{\mathcal{X},\log}$ and whose differentials are certain Bockstein homomorphisms.

Each $\widehat{\Omega}^i_{\mathcal{X},\log}$ is a formal vector bundle on $\mathfrak{X}$, each $\mathcal{X}_{\mathcal{O}/p^n}$ has no embedded associated primes, and $\mathfrak{X}^{\mathrm{sm}}$ is dense in $\mathfrak{X}$, so the agreement of the differentials with the differentials of $\widehat{\Omega}^\bullet_{\mathcal{X},\log}$ may be checked over $\mathfrak{X}^{\mathrm{sm}}$ where it follows from Remark 2.26 and [BMS16, 14.1 (ii)] (instead of loc. cit. one may also use [Bha16, proof of Prop. 7.9]). ☐

4. The construction of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ and the analysis of its cohomology

With the results of §§2–3 in hand, we are ready to define the promised A_{inf}-cohomology groups (see §4.2) and to detail some of their properties. We give the comparisons to the logarithmic de Rham and to the p-adic étale cohomology in Theorems 4.4 and 4.5, and we prove cohomology specialization results in Theorem 4.12 and Corollary 4.13. Granted the inputs from §§2–3, the arguments are similar to those in the smooth case treated in [BMS16]; the key difference is that we have to find ways to bypass crystalline inputs (cf. Remark 4.7). For instance, in the case of Corollary 4.13 this means that the conclusion is slightly weaker than its analogue proved in the good reduction case in [BMS16, 14.4 and 14.5 (ii)].

4.1. Properness of $\mathcal{X}$. In §4 we assume that $\mathcal{X}$ is proper and of pure relative dimension d over $\mathcal{O}$.

4.2. The A_{inf}-cohomology $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$. Using $A\Omega_{\mathfrak{X}} \in D^{\geqslant 0}(\mathfrak{X}_{\text{ét}}, A_{\mathrm{inf}})$ defined in §3.2, we set

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) := R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \in D^{\geqslant 0}(A_{\mathrm{inf}}) \quad \text{and} \quad H^i_{A_{\mathrm{inf}}}(\mathcal{X}) := H^i(R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})) \quad \text{for } i \in \mathbb{Z}.$$

By Lazard's theorem and [SP, 0739] (or [SGA $4_{\rm II}$, VI, 5.2]), $R\Gamma_{A_{\rm inf}}(\mathcal{X})\otimes_{A_{\rm inf}} A \cong R\Gamma(\mathfrak{X}_{\rm \acute{e}t}, A\Omega_{\mathfrak{X}}\otimes_{A_{\rm inf}} A)$ for every flat $A_{\rm inf}$-module A, so, in particular, (3.2.1) gives an isomorphism

$$(R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes_{A_{\rm inf},\varphi} A_{\rm inf})[\tfrac{1}{\tilde{\xi}}] \cong (R\Gamma_{A_{\rm inf}}(\mathcal{X}))[\tfrac{1}{\tilde{\xi}}]. \tag{4.2.1}$$

We begin detailing properties of $R\Gamma_{A_{\rm inf}}(\mathcal{X})$ by identifying its de Rham specialization in Theorem 4.4.

4.3. The notation $H^i_{\log\,\mathrm{dR}}(\mathcal{X}/\mathcal{O})$. For the rest of §4, we let $\Omega^\bullet_{\mathcal{X},\,\log}$ be the logarithmic de Rham complex of $\mathcal{X}$ defined in Theorem 3.15 and for an $i \in \mathbb{Z}$ we set

$$H^i_{\log\,\mathrm{dR}}(\mathcal{X}/\mathcal{O}) := R^i\Gamma(\mathcal{X}, \Omega^\bullet_{\mathcal{X},\,\log}).$$

Theorem 4.4. *We have an identification*

$$R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\rm inf},\,\theta} \mathcal{O}_C \cong R\Gamma(\mathcal{X}, \Omega^\bullet_{\mathcal{X},\,\log}) \otimes_{\mathcal{O}} \mathcal{O}_C.$$

Proof. By the projection formula [SP, 0944],

$$R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\rm inf},\,\theta} \mathcal{O}_C \cong R\Gamma(\mathfrak{X}_{\rm \acute{e}t}, A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\rm inf},\,\theta} \mathcal{O}_C).$$

Moreover, by Theorem 3.15, $A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\rm inf},\,\theta} \mathcal{O}_C \cong \widehat{\Omega}^\bullet_{\mathcal{X},\,\log}$, so we have the E_1-spectral sequences

$$H^j(\mathcal{X}, \Omega^i_{\mathcal{X},\,\log})_{\mathcal{O}_C} \Rightarrow H^{i+j}(R\Gamma(\mathcal{X}, \Omega^\bullet_{\mathcal{X},\,\log}))_{\mathcal{O}_C} \quad \text{and} \quad H^j(\mathfrak{X}, \widehat{\Omega}^i_{\mathcal{X},\,\log}) \Rightarrow H^i(R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\rm inf},\,\theta} \mathcal{O}_C).$$

By the Grothendieck comparison theorem in this context [FK14, I.9.2.1] (cf. also Theorem 2.23),

$$H^j(\mathcal{X}, \Omega^i_{\mathcal{X},\,\log})_{\mathcal{O}_C} \cong H^j(\mathfrak{X}, \widehat{\Omega}^i_{\mathcal{X},\,\log}) \qquad \text{for all } i, j,$$

so the natural map from the first spectral sequence to the second one is an isomorphism. Consequently,

$$R\Gamma(\mathcal{X}, \Omega^\bullet_{\mathcal{X},\,\log}) \otimes_{\mathcal{O}} \mathcal{O}_C \cong R\Gamma(\mathfrak{X}, \widehat{\Omega}^\bullet_{\mathcal{X},\,\log}).$$ ☐

As the following theorem shows, the étale specialization behaves precisely as in the smooth case.

Theorem 4.5. *Setting $R\Gamma(\mathcal{X}_{C,\,\rm \acute{e}t}, \mathbb{Z}_p) := R\lim_n(R\Gamma(\mathcal{X}_{C,\,\rm \acute{e}t}, \mathbb{Z}/p^n\mathbb{Z}))$, we have ($\mu$ was defined in §3.1)*

$$R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes_{A_{\rm inf}} A_{\rm inf}[\tfrac{1}{\mu}] \cong R\Gamma(\mathcal{X}_{C,\,\rm \acute{e}t}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\rm inf}[\tfrac{1}{\mu}].$$

Proof. By [BMS16, 6.14], $A\Omega_{\mathfrak{X}} \otimes_{A_{\rm inf}} A_{\rm inf}[\frac{1}{\mu}] \cong R\nu_* \mathbb{A}_{{\rm inf},\,X} \otimes_{A_{\rm inf}} A_{\rm inf}[\frac{1}{\mu}]$, so

$$R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes_{A_{\rm inf}} A_{\rm inf}[\tfrac{1}{\mu}] \cong R\Gamma(X^{\rm ad}_{\rm pro\acute{e}t}, \mathbb{A}_{{\rm inf},\,X}) \otimes_{A_{\rm inf}} A_{\rm inf}[\tfrac{1}{\mu}].$$

Moreover, by [BMS16, 5.6],

$$R\Gamma(X^{\rm ad}_{\rm pro\acute{e}t}, \mathbb{A}_{{\rm inf},\,X}) \otimes_{A_{\rm inf}} A_{\rm inf}[\tfrac{1}{\mu}] \cong R\Gamma(X^{\rm ad}_{\rm \acute{e}t}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\rm inf}[\tfrac{1}{\mu}].$$

It remains to apply [Hub96, 3.7.2] to obtain $R\Gamma(X^{\rm ad}_{\rm \acute{e}t}, \mathbb{Z}_p) \cong R\Gamma(\mathcal{X}_{C,\,\rm \acute{e}t}, \mathbb{Z}_p)$. ☐

We turn to some structural properties of the cohomology object $R\Gamma_{A_{\rm inf}}(\mathcal{X})$, such as its perfectness.

Proposition 4.6. *The object $R\Gamma_{A_{\rm inf}}(\mathcal{X}) \in D^{\geqslant 0}(A_{\rm inf})$ is perfect, i.e., it is isomorphic to a bounded complex of finite free $A_{\rm inf}$-modules. The $A_{\rm inf}$-module $H^i_{A_{\rm inf}}(\mathcal{X})$ vanishes if $i \notin [0, 2d]$.*

Proof. By Theorem 4.4 and [SP, 066U], $R\Gamma_{A_{\rm inf}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\rm inf}} A_{\rm inf}/(\xi)$ is a perfect complex of $\mathcal{O}_C$-modules (with ξ defined in §3.1). Moreover, by Proposition 3.14 (b), $A\Omega_{\mathfrak{X}}$, and hence also $R\Gamma_{A_{\rm inf}}(\mathcal{X})$, is derived ξ-adically complete. Therefore, by [SP, 09AW], $R\Gamma_{A_{\rm inf}}(\mathcal{X})$ is perfect, which gives the first claim. The top degree cohomology of $R\Gamma_{A_{\rm inf}}(\mathcal{X})$ is then finitely presented and its formation commutes with base change. Thus, by the Nakayama lemma and Theorem 3.15, $H^i_{A_{\rm inf}}(\mathcal{X}) = 0$ for $i > 2d$. ☐

Remark 4.7. We expect that each $H^i_{A_{\inf}}(\mathcal{X})$ is finitely presented and even perfect as an $A_{\inf}$-complex. The proof of this for smooth $\mathcal{X}$ uses the crystalline specialization, see [BMS16, 4.9, 4.20].

In the following case the analysis of individual $H^i_{A_{\inf}}(\mathcal{X})$ does not require the crystalline specialization.

Proposition 4.8. *If $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})$ is p-torsion free for $i \geqslant i_0$ and some $i_0 \in \mathbb{Z}$, then $H^i_{A_{\inf}}(\mathcal{X})$ is finite free over $A_{\inf}$ for $i \geqslant i_0$ and*

$$H^i_{A_{\inf}}(\mathcal{X}) \otimes_{A_{\inf},\theta} \mathcal{O}_C \xrightarrow{\sim} H^i(R\Gamma_{A_{\inf}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\inf},\theta} \mathcal{O}_C) \overset{4.4}{\cong} H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})_{\mathcal{O}_C} \quad \text{for } i \geqslant i_0 - 1. \quad (4.8.1)$$

In particular, the map in (4.8.1) is always an isomorphism for $i = 2d$.

The proof will use the following lemma.

Lemma 4.9. *If $(R, \mathfrak{m})$ is a local domain, then a finite R-module M is free if and only if*

$$\dim_{R/\mathfrak{m}R}(M/\mathfrak{m}M) = \dim_{\mathrm{Frac}(R)}(M_{\mathrm{Frac}(R)}). \quad (4.9.1)$$

Proof. By the Nakayama lemma, a lift $m_1, \dots, m_d \in M$ of an $R/\mathfrak{m}R$-basis of $M/\mathfrak{m}M$ generates M, and hence also contains a basis of $M_{\mathrm{Frac}(R)}$. Thus, if (4.9.1) holds, then the m_i can have no R-relation, and hence must define an isomorphism $R^d \simeq M$. The converse is clear. ☐

Proof of Proposition 4.8. We will argue by decreasing induction on i_0. Proposition 4.6 supplies the case $i_0 > 2d + 1$, so suppose that $H^i_{A_{\inf}}(\mathcal{X})$ is finite free for $i \geqslant i_0 + 1$. Then, by Proposition 4.6 and [SP, 066U and 066R], $\tau_{\leqslant i_0} R\Gamma_{A_{\inf}}(\mathcal{X}) \in D^{[0,i_0]}(A_{\inf})$ is perfect, so the formation of its cohomology in degree i_0 commutes with arbitrary base change and the $A_{\inf}$-module $H^{i_0}_{A_{\inf}}(\mathcal{X})$ is finitely presented. Moreover, $\tau_{\geqslant i_0+1} R\Gamma_{A_{\inf}}(\mathcal{X})$ has $A_{\inf}$-free cohomology, so the formation of this cohomology commutes with arbitrary base change. Thus, the triangle

$$(\tau_{\leqslant i_0} R\Gamma_{A_{\inf}}(\mathcal{X})) \otimes^{\mathbb{L}}_{A_{\inf},\theta} \mathcal{O}_C \to R\Gamma_{A_{\inf}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\inf},\theta} \mathcal{O}_C \to (\tau_{\geqslant i_0+1} R\Gamma_{A_{\inf}}(\mathcal{X})) \otimes^{\mathbb{L}}_{A_{\inf},\theta} \mathcal{O}_C \xrightarrow{[1]}$$

shows that (4.8.1) holds for $i \geqslant i_0$. In particular, if $H^{i_0}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})$ is p-torsion free, then

$$\dim_{\overline{k}}(H^{i_0}_{A_{\inf}}(\mathcal{X}) \otimes_{A_{\inf}} \overline{k}) = \dim_k(H^{i_0}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}) \otimes_{\mathcal{O}} k) = \dim_K H^{i_0}_{\mathrm{dR}}(\mathcal{X}_K/K).$$

Moreover, $\dim_K H^{i_0}_{\mathrm{dR}}(\mathcal{X}_K/K) = \dim_{\mathbb{Q}_p} H^{i_0}_{\text{ét}}(\mathcal{X}_C, \mathbb{Q}_p)$ and, by Theorem 4.5, the latter equals the $\mathrm{Frac}(A_{\inf})$-dimension of $H^{i_0}_{A_{\inf}}(\mathcal{X}) \otimes_{A_{\inf}} \mathrm{Frac}(A_{\inf})$. In conclusion, if $H^{i_0}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})$ is p-torsion free, then $H^{i_0}_{A_{\inf}}(\mathcal{X})$ is finite free by Lemma 4.9, which completes the inductive step. ☐

We will phrase the cohomology specialization Theorem 4.12 in terms of he following formalism.

4.10. The formalism of normalized length. Let $\mathfrak{o}$ be a rank 1 valuation ring of mixed characteristic $(0, p)$ and normalize its valuation $\mathrm{val}_{\mathfrak{o}}$ by requiring that $\mathrm{val}_{\mathfrak{o}}(p) = 1$. By the structure theorem (cf. [GR03, 6.1.14] or [SP, 0ASP]), every finitely presented torsion $\mathfrak{o}$-module M is of the form

$$M \cong \textstyle\bigoplus_{i=1}^n \mathfrak{o}/(a_i) \quad \text{for some} \quad a_i \in \mathfrak{o}\backslash\{0\}, \quad \text{and we set} \quad \mathrm{val}_{\mathfrak{o}}(M) := \sum_{i=1}^n \mathrm{val}(a_i).$$

In other words, $\mathrm{val}_{\mathfrak{o}}(M)$ is the valuation of any generator of the 0^{th} Fitting ideal $\mathrm{Fitt}_0(M) \subset \mathfrak{o}$ of M, so it depends only on M. If $\mathfrak{o}$ is a discrete valuation ring for which p is a uniformizer, then $\mathrm{val}_{\mathfrak{o}}(M) = \mathrm{length}_{\mathfrak{o}}(M)$. In general, $\mathrm{val}_{\mathfrak{o}}$ has the advantage of being invariant under extension of scalars to a larger $\mathfrak{o}$. Any short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of finitely presented torsion $\mathfrak{o}$-modules gives rise to the equality $\mathrm{Fitt}_0(M_2) = \mathrm{Fitt}_0(M_1)\,\mathrm{Fitt}_0(M_3)$ (cf. [GR03, 6.3.1 and 6.3.5 (i)]), so the assignment $\mathrm{val}_{\mathfrak{o}}(-)$ satisfies

$$\mathrm{val}_{\mathfrak{o}}(M_2) = \mathrm{val}_{\mathfrak{o}}(M_1) + \mathrm{val}_{\mathfrak{o}}(M_3). \tag{4.10.1}$$

The following variant of [BMS16, 4.14] is important for the proof of the subsequent Theorem 4.12.

Lemma 4.11. *For an $n \in \mathbb{Z}_{\geqslant 1}$, a finitely presented $W_n(\mathcal{O}_C^\flat)$-module M, and ξ as in §3.1,*

$$\mathrm{val}_{W(C^\flat)}(M \otimes_{A_{\mathrm{inf}}} W(C^\flat)) = \mathrm{val}_{\mathcal{O}_C}(M/\xi M) - \mathrm{val}_{\mathcal{O}_C}(M[\xi]). \tag{4.11.1}$$

Proof. Since $W_n(\mathcal{O}_C^\flat)$ is a coherent ring (cf. [BMS16, 3.24]), $M[\xi]$ is finitely presented over $W_n(\mathcal{O}_C^\flat)$. Moreover, due to (4.10.1), the flatness of $A_{\mathrm{inf}} \to W(C^\flat)$ (the localization of A_{inf} at pA_{inf} is a discrete valuation ring whose completion identifies with $W(C^\flat)$, cf. [BMS16, proof of Lem. 4.10]), and the snake lemma, both sides of (4.11.1) are additive in short exact sequences. We may therefore assume that $n = 1$ and, due to the structure theorem [SP, 0ASP], that $M = \mathcal{O}_C^\flat/(x)$ for some $x \in \mathcal{O}_C^\flat$.

If $x = 0$, then both sides of (4.11.1) are equal to 1. If $x \neq 0$, then the left side vanishes, and so does the right side because $M[\xi] \cong \mathrm{Tor}^1_{\mathcal{O}_C^\flat}(M, \mathcal{O}_C/p)$ and the following sequence is exact:

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_C^\flat}(\mathcal{O}_C^\flat/(x), \mathcal{O}_C/p) \to \mathcal{O}_C/p \xrightarrow{\theta([x])} \mathcal{O}_C/p \to M/\xi M \to 0.$$ ☐

Theorem 4.12. *For $i \in \mathbb{Z}$, letting $H^i_{\log \mathrm{dR}}(\mathcal{X}_{\mathcal{O}/p^n})$ denote the i^{th} hypercohomology group of the complex $\Omega^\bullet_{\mathcal{X}_{\mathcal{O}/p^n}, \log} := (\Omega^\bullet_{\mathcal{X}, \log})_{\mathcal{O}/p^n}$, we have*

$$\mathrm{val}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n\mathbb{Z})) \leqslant \mathrm{val}_{\mathcal{O}}(H^i_{\log \mathrm{dR}}(\mathcal{X}_{\mathcal{O}/p^n})) \qquad \text{for all } n \in \mathbb{Z}_{\geqslant 0}. \tag{4.12.1}$$

Proof. By Theorems 4.4 and 4.5, $B := R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W_n(\mathcal{O}_C^\flat) \in D(W_n(\mathcal{O}_C^\flat))$ satisfies (for the second isomorphism see also the last aspect of [SP, 09AW])

$$\begin{aligned} B \otimes^{\mathbb{L}}_{W_n(\mathcal{O}_C^\flat), \bar\theta} \mathcal{O}_C/p^n &\cong R\Gamma(\mathcal{X}_{\mathcal{O}/p^n}, (\Omega^\bullet_{\mathcal{X}, \log})_{\mathcal{O}/p^n}) \otimes_{\mathcal{O}/p^n} \mathcal{O}_C/p^n, \\ B \otimes_{W_n(\mathcal{O}_C^\flat)} W_n(C^\flat) &\cong R\Gamma(\mathcal{X}_{C, \text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} W_n(\mathcal{O}_C^\flat). \end{aligned} \tag{4.12.2}$$

The ring $W_n(\mathcal{O}_C^\flat)$ is coherent (cf. [BMS16, 3.24]) and, by Proposition 4.6, B is perfect, so the cohomology groups of B are finitely presented. In particular, Lemma 4.11 and (4.12.2) imply that

$$\mathrm{val}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n\mathbb{Z})) = \mathrm{val}_{W(C^\flat)}(H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} W(C^\flat)) \leqslant \mathrm{val}_{\mathcal{O}_C}(H^i(B) \otimes_{W_n(\mathcal{O}_C^\flat), \bar\theta} \mathcal{O}_C/p^n).$$

Since ξ is not a zero divisor in $W_n(\mathcal{O}_C^\flat)$, it remains to combine (4.12.2) with a spectral sequence from [SP, 0662] (or with a suitable variant of the argument used in the proof of Proposition 4.6) to get the injection

$$H^i(B) \otimes_{W_n(\mathcal{O}_C^\flat), \bar\theta} \mathcal{O}_C/p^n \hookrightarrow H^i_{\log \mathrm{dR}}(\mathcal{X}_{\mathcal{O}/p^n}) \otimes_{\mathcal{O}/p^n} \mathcal{O}_C/p^n.$$ ☐

Corollary 4.13. *If $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})$ and $H^{i+1}_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})$ are p-torsion free, then so is $H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p)$.*

Proof. The assumptions ensure that

$$H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}/p \xrightarrow{\sim} H^i_{\log \mathrm{dR}}(\mathcal{X}_{\mathcal{O}/p}).$$

Thus, since

$$H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z} \hookrightarrow H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p\mathbb{Z})$$

and the $\mathbb{Z}_p$-rank of $H^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p)$ equals the $\mathcal{O}$-rank of $H^i_{\log \mathrm{dR}}(\mathcal{X}/\mathcal{O})$, (4.12.1) gives the conclusion. ☐

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