Geometric & abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

- Equivariant Transversality in Geometric Regularization
- Guiding Questions for studying regularization approaches

  more basic than

- How to deal with isotropy

- Nontransverse Gluing
  (by stabilization)

Geometric & abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

Example:

$A \neq 0 \in H_2(M)$ s.t. $\overline{\mathcal{M}}$ compact

$$\overline{\mathcal{M}} = \{u : \mathbb{P}^1 \to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1] = A\} / u \sim u \circ \varphi \quad \forall \varphi\ G(\mathbb{P}^1, i)\ \ \varphi(\infty) = \infty$$

$\downarrow ev$ $\quad$ $\mapsto$

M $\quad$ $u(\infty)$

Goals:

* $\overline{\mathcal{M}}^{reg}$ cobordism class of (smooth) manifolds resp. orbifolds/weighted branched manifolds "moduli cycle"

  expect $\mathbb{Z}$ resp $\mathbb{Q}$ coefficient

* $[\overline{\mathcal{M}}] \in H_*(\overline{\mathcal{M}})$ "moduli class"

* $ev_*[\overline{\mathcal{M}}] \in H_*(M)$ "virtual moduli class"

$\boxed{\text{Geometric Regularization}}$	$\boxed{\text{Abstract Regularization}}$

$\overline{\mathcal{M}}_J = \bar\partial_J^{-1}(0)/\mathrm{Aut} \cup_{\text{glue}} \mathcal{M}^{\text{broken}} \rightsquigarrow \overline{\mathcal{M}}_{J'} = \overline{\mathcal{M}}^{\text{reg}}$

0.) Global Fredholm property of $\bar\partial_J$

1.) Equivariant Transversality
$\exists J' : \bar\partial_{J'} \pitchfork 0$ ⊛

2.) Quotient Theorem
$\bar\partial_J^{-1}(0) \supsetneq \mathrm{Aut}$

3.) Gromov Compactness & Gluing
$\overline{\mathcal{M}}_{J'} := \bar\partial_{J'}^{-1}(0)/\mathrm{Aut} \cup_{\text{glue}_J} \mathcal{M}_{J'}^{\text{broken}}$ compact

4.) Cobordism / Continuation map
$\forall J'' \neq J' \;\; \exists$ cobordism $\overline{\mathcal{M}}_{J''} \sim \overline{\mathcal{M}}_{J'}$

(in more algebraic theories (e.g. Floer homology)
* $\exists$ chain maps $(CF,\partial)_{J''} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} (CF,\partial)_{J'}$
* $\exists$ chain homotopy equivalence $\varphi\circ\psi \sim id$)

$\overline{\mathcal{M}} = s^{-1}(0) \rightsquigarrow (s+\nu)^{-1}(0) = \overline{\mathcal{M}}^{\text{reg}}$

0.) Gromov Compactness & Gluing Topology

1.) local Fredholm descriptions

1'.) transition information

2.) regularization theorem:
$\exists \nu : s+\nu \pitchfork 0$, ...

⊛ i.e. $\underset{\bar\partial_J}{\underset{\shortparallel}{s}} + \underset{\bar\partial_{J'} - \bar\partial_J}{\underset{\shortparallel}{\nu}} \pitchfork 0$, $\varphi_* \nu = \nu \circ \varphi_*$ $\forall \varphi \in \mathrm{Aut}$

Equivant Transversality in Geometric Regularization

$\overline{\mathcal{M}}_J = \bar\partial_J^{-1}(0) / \mathrm{Aut} \;\cup\; \mathcal{M}^{\text{broken}}$ (glue)

- space $\mathcal{P}$ of Aut-equivariant perturbations

  $\forall p \in \mathcal{P} : p \circ \varphi^* = \varphi^* \circ p \;\; \forall \varphi \in \mathrm{Aut}$ e.g. $\mathcal{P} = \{ \bar\partial_{J'} - \bar\partial_J \mid J' \in \mathcal{J}(M,\omega) \}$

- transversality for universal moduli space $\mathcal{M}_{\mathcal{P}} := \{ (u,p) \mid (\bar\partial_J + p)(u) = 0 \}$

  i.e. $(u,p) \longmapsto (\bar\partial_J + p)(u) \pitchfork 0$

  i.e. $(\xi, P) \longmapsto \underbrace{D_u \bar\partial_J \xi + D_u p\, \xi}_{D_u \bar\partial_{J'}} + P(u)$ right invertible

  i.e. (at $p = 0$ using Lemma): $0 \neq \eta \perp \operatorname{im} D_u \bar\partial_J \Rightarrow \exists P \in T\mathcal{P} : \langle \eta, P(u) \rangle \neq 0$

- **Sard-Smale Theorem:** $\pi : \mathcal{M}_{\mathcal{P}} \to \mathcal{P}$, $(u,p) \mapsto p$, $\mathcal{C}^k$, Fredholm between $\mathcal{C}^k$-Banach manifolds

  $k \geq \operatorname{index} d\pi + 1$

  $\Rightarrow \mathcal{P}^{\text{reg}} = \{ p \in \mathcal{P} \mid \forall (u,p) \in \mathcal{M}_{\mathcal{P}} : d_{(u,p)}\pi \text{ onto} \} \subset \mathcal{P}$ comeagre ($\Rightarrow$ dense)

  $u \in (\bar\partial_J + p)^{-1}(0)$ $D_u \bar\partial_J$ onto

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- **Fredholm Lemma:** $D_u \bar\partial_J$ Fredholm, $E_u : P \mapsto P(u)$ bounded, $\operatorname{im}(D_u \bar\partial_J + E_u)$ dense

  $\Rightarrow D_u \bar\partial_J + E_u$ right invertible, $\Pi : \ker(D_u \bar\partial_J + E_u) \to T\mathcal{P}$ Fredholm

  $d\pi : T_{(u,p)} \mathcal{M}_{\mathcal{P}}$

  $\ker \Pi \simeq \ker D_u \bar\partial_J$, $\operatorname{coker} \Pi \simeq \operatorname{coker} D_u \bar\partial_J \Rightarrow \operatorname{ind} \underset{d\pi}{\Pi} = \operatorname{ind} D_u \bar\partial_J \leq k-1$

Equivariant Transversality in Geometric Regularization

- find Aut-equiv perturbations $\mathcal{P}$

note: $p = \bar\partial_{J'} - \bar\partial_J$ is 1st order, so allow "local p" to depend on germ of u

"local" case: $p : \Sigma \times M \to \dots$

$p(u) : z \mapsto p(z, u(z))$ ⊛

$p(u\circ\varphi) = p(u)\circ d\varphi$

$p(z, u(\varphi(z))) = p(\varphi(z), u(\varphi(z)))\circ d\varphi$

$\Rightarrow$ $p \in \mathcal{P}$ must be "invariant along Aut-orbits"

$\{\varphi(z) \mid \varphi \in Aut\}$

i.e. $p(u)(s,t) = p(t, u(s,t))$ for $z = (s,t) \in \Sigma \simeq Aut\cdot z \times T$

e.g. $\mathbb{R} \curvearrowright$ cylinder $\simeq \mathbb{R}\times S^1$

$\{z \mapsto az+b\} \curvearrowright \mathbb{P}^1$ $\simeq \mathbb{P}^1 \times$ point

- give $\bar\partial_J : \mathcal{B} \to \mathcal{E}$, $\mathcal{P}$ $\mathcal{C}^k$-differentiability, $k \geq \operatorname{ind} D\bar\partial_J + 1$
- $0 \neq \eta \perp \operatorname{im} D_u\bar\partial_J \Rightarrow \exists P \in T\mathcal{P} : \langle \eta, P(u)\rangle \neq 0$

$$\langle \eta, P(u)\rangle = \int_\Sigma \langle \eta(z), P(z, u(z))\rangle = \int_T \langle \eta(\cdot, t), P(t, u(\cdot,t))\rangle_{L^2(Aut\cdot z)}\, dt$$

$\Downarrow$

$D^*\eta = 0 \Rightarrow \eta \in \mathcal{C}^\infty$

$\exists (s_0, t_0) : \eta(s_0,t_0) \neq 0$

$P = \text{cutoff}(t)\cdot P(t_0, \cdot)$

- find t_0, $P(t_0, \cdot)$ s.t. $\int_{Aut\cdot z} \langle \eta(\cdot, t_0), P(t_0, u(\cdot, t_0))\rangle \neq 0$

$P(t_0,\cdot) = \beta \cdot P_0$

$\int_{Aut\cdot z} \langle \eta(\cdot, t_0), P(t_0, u(\cdot, t_0))\rangle = \int \langle \eta(s, t_0), \underbrace{\beta(u(s,t_0))}_{\text{supported in } \{s \approx s_0\}} P_0\rangle$

(i) find P_0 s.t. $\langle \eta(s_0,t_0), P_0\rangle > 0$

$\Downarrow$

$\forall s \approx s_0$ $\langle \eta(s,t_0), P_0\rangle > 0$

(ii) find t_0 s.t. $u(s,t_0) = u(s_0,t_0) \Rightarrow s \approx s_0$

requires "somewhere injectivity": for almost all t_0 $\exists\, s_0$:

(i) $\partial_s u(s_0) \neq 0$ (ii) $u(s,t_0) \neq u(s_0,t_0)$ $\forall s \neq s_0$

since $P = \hat{P}\cdot \partial_s u$

Guiding Questions for studying regularization approaches

& "beware of ..."

- via equivariant transversality (with nondiscrete group) → what perturbations?
  ? compatibility with Gromov compactness ?
  → why is universal linearized operator surjective?
  ? require "somewhere injectivity" of curves ?

- via abstract perturbations / "virtual" fundamental class

  $\overline{\mathcal{M}} = s^{-1}(0)$ $\quad [(s+\nu)^{-1}(0)]$ / Euler(s) $= [\overline{\mathcal{M}}]$

  → what is the abstract form of section s? ⚠ fuzzy notation
  ↳ e.g. when $\bar\partial_J \pitchfork 0$?

  → why does regularization theorem hold? ⚠ topology, e.g. Hausdorff compactness ((overly simplified intuition))

  → how is s constructed for pseudoholomorphic curve moduli spaces? ⚠ analysis, e.g. reparametrization
  ↳ from local Fredholm descriptions
  ↳ in basic example $\overline{\mathcal{M}}_{0,1}(A,J) = \dfrac{\{u : \mathbb{P}^1 \to M \mid u_*[\mathbb{P}^1] = A,\ \bar\partial_J u = 0\}}{\{\varphi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1,\ \varphi(\infty) = \infty\}}$