

Analytic description of moduli spaces,

Geometric Regularization

- "transverse gluing" — local surjectivity

Abstract Regularization

- why?
- Fredholm stabilization
- "stabilized gluing"

Geometric Regularization

(Ex: Hamiltonian Floer without bubbling)

0.) Global Fredholm description of $\tilde{M} = \bar{\partial}_{J,H}^{-1}(0)$

1.) Equivariant Transversality

find J s.t. $\begin{matrix} \Sigma \\ \downarrow \\ \mathcal{B} = \mathcal{C}^{-1}(R \times S^1, M) \end{matrix} \bar{\partial}_{J,H} \not\equiv 0 \Rightarrow \bar{\partial}_{J,H}^{-1}(0) = \mathcal{B}$
smooth submanifold

2.) Quotient Theorem

$\mathbb{R} \curvearrowright \bar{\partial}_{J,H}^{-1}(0)$ smooth, proper, free $\Rightarrow M = \frac{\bar{\partial}_{J,H}^{-1}(0)}{\mathbb{R}}$ smooth manifold
 $= \bigcup_{k \geq 0} M^k \quad \dim M^k = k$

3.) Gromov Compactness & Gluing

$\mathcal{S} : M_{\text{per}}^0 \times M^0 \times (R, \infty) \rightarrow M^1$ (1) injective
(2) locally surjective: $[u_i] \xrightarrow{i \rightarrow \infty} ([u_-], [u_+])$
 $\Rightarrow \tilde{M}^1 = M^1 \cup_{\mathcal{S}} M_{\text{per}}^0 \times M^0 \times (R, \infty)$ compact 1-mfd with $\partial \tilde{M}^1 \simeq M_{\text{per}}^0 \times M^0$
 $\Rightarrow \forall i \geq i_0 \quad [u_i] \in \text{im } \mathcal{S}$

4.) Cobordism / Continuation map

for $J_0 \neq J_1$ construct $(CF, \partial_0) \simeq (CF, \partial_1)$ from $M((J_t)_{t \in [0,1]})$
 $\Theta \rightarrow$ steps 1-3 twice more [see Salamon script]

local surjectivity: $[u_i] \xrightarrow{i \rightarrow \infty} ([u_-], [u_+]) \Rightarrow \forall i \geq i_0, [u_i] \in \text{img}$

$$\begin{pmatrix} u_i(\cdot - s_i, \cdot) \rightarrow u_- \\ u_i(\cdot + s_i, \cdot) \rightarrow u_+ \\ E(u_-) + E(u_+) = E(u_i) = E \end{pmatrix} \begin{matrix} \uparrow \\ \text{choice } s_i \rightarrow \infty \end{matrix}$$

$$\int_{-k}^k |\partial_r u_-|^2 + \int_{-k}^k |\partial_r u_+|^2 \approx \int_{-k-s_i}^{k-s_i} |\partial_r u_i|^2 + \int_{-k+s_i}^{k+s_i} |\partial_r u_i|^2 \geq E - \varepsilon_k$$

$\forall i \geq I_k$

\uparrow Marton iteration

$$u_i(\cdot - \varepsilon, \cdot) = \exp_{u_- \#_{R_i} u_+} \zeta$$

$\zeta \in \text{im } D_{R_i}^+, \|\zeta\|_{W^1, p} < \Delta$

\uparrow IFT $\exists r_i, \varepsilon_i$
 $\zeta_{r_i, \varepsilon_i} \in \text{im } D_{s_i+r_i}^+$

$$u_i = \exp_{u_- \#_{s_i} u_+} \zeta_i$$

$$\|\zeta_i\|_{W^1, p} \leq C \left(\begin{matrix} d_{(-k, k)}(u_i(\cdot - s_i), u_-) \\ + d_{(-k, k)}(u_i(\cdot + s_i), u_+) \end{matrix} \right) \begin{matrix} \xrightarrow{i \rightarrow \infty} 0 \\ \text{fix } k \end{matrix}$$

$$+ d_{(k-s_i, -k+s_i)}(u_i, \gamma) + d_{(-\infty, -k-s_i)}(u_i, \gamma_-) + d_{(-\infty, -k)}(\gamma_-, u_-) + d_{(-\infty, -k)}(\gamma, \exp_{u_+}(\beta_2))$$

$$+ d_{(k+s_i, \infty)}(u_i, \gamma_+) + d_{(k, \infty)}(\gamma_+, u_+) + d_{(k, \infty)}(\gamma, \exp_{u_-}(\beta_2))$$

$\leq C \varepsilon_k$

$$\Rightarrow u_i(\cdot + \varepsilon_i) = \exp_{u_- \#_{s_i+r} u_+} \zeta_{r, \varepsilon}$$

$$\|\zeta_{r, \varepsilon}\| < \Delta \quad \forall |r|, |\varepsilon| \leq \delta_0$$

$< \Delta/2$ by choice of $k, i \geq I_{k, \Delta}$

IFT: $f(r, \varepsilon) := \begin{pmatrix} \langle \zeta_{r, \varepsilon}, \alpha_r^- \rangle \\ \langle \zeta_{r, \varepsilon}, \alpha_r^+ \rangle \end{pmatrix} \left(\text{span}(\alpha_r^-, \alpha_r^+) = \ker D_{u_- \#_{s_i+r} u_+} \bar{\partial}_{\beta, \mathcal{M}} \right)^\perp = \text{im } D_{s_i+r}^+$

$$\zeta_{r, \varepsilon} \approx u_i(\cdot + \varepsilon) - u_i(\cdot - s_i - r) - u_+(\cdot + s_i + r) \quad \alpha_r^- \approx \partial_s u_-(\cdot - s_i - r, \cdot) \quad \langle \alpha_r^-, \alpha_r^+ \rangle \approx 0$$

$$\Rightarrow \partial_\varepsilon \zeta_{r, \varepsilon} \Big|_{\varepsilon=0} \approx \partial_s u_i \approx \alpha_0^- + \alpha_0^+ \quad \alpha_r^+ \approx \partial_s u_+(\cdot + s_i + r, \cdot)$$

$$\partial_r \zeta_{r, \varepsilon} \Big|_{\varepsilon=0} \approx \partial_s u_-(\cdot - s_i) - \partial_s u_+(\cdot + s_i) = \alpha_0^- - \alpha_0^+$$

$$\zeta_{r, \varepsilon} \approx 0 \Rightarrow df(0, 0) \approx \begin{pmatrix} \langle \alpha^- - \alpha^+, \alpha^- \rangle, \langle \alpha^- + \alpha^+, \alpha^- \rangle \\ \langle \alpha^- - \alpha^+, \alpha^+ \rangle, \langle \alpha^- + \alpha^+, \alpha^+ \rangle \end{pmatrix} \approx \begin{pmatrix} |\alpha^-|^2 & |\alpha^-|^2 \\ -|\alpha^+|^2 & |\alpha^+|^2 \end{pmatrix} \text{ invertible}$$

Abstract Regularization

Reason: regular J rarely exist

Tool: generalized regularization theorem for sections with $s^{-1}(0)$ compact

Approach: $\bar{M} = s^{-1}(0)$

0.) quotient & compactify (glue $\bar{M} = \tilde{M}_{\text{Ant}} \cup M^{\text{broken}}$)

1.) local Fredholm descriptions $\bar{M} = \cup F_i$ @: near M^{broken} ?

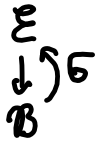
$\begin{matrix} \Sigma_i \\ \downarrow \uparrow \\ \mathcal{B}_i \end{matrix}$ s : Fredholm section, $s_i^{-1}(0) \xrightarrow[\text{homeom}]{\cong} F_i \subset \bar{M}$
open

1') transition information $\bar{M} = \cup F_i / \sim = "s^{-1}(0)"$

2.) regularization theorem: $\exists \{r\}: s + r \neq 0$ (and $(s+r)^{-1}(0)$ mfd)

$\bar{M}^r := (s+r)^{-1}(0)$ compact, $\partial \bar{M}^r = \bar{M}^r \times \bar{M}^r$
unique up to cobordism

local Fredholm description

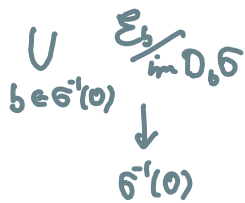


Fredholm section

$\sigma^{-1}(0)$
finite symmetry group

$\hookrightarrow \bar{M}$
local homeom.

obstruction "bundle"



stabilization

$$E \subset \mathcal{E}|_{\text{nbhd}(\sigma^{-1}(0))}$$

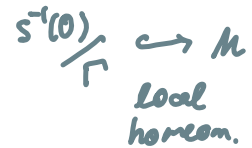
$$E_b + \text{in } D_b \sigma = \mathcal{E}_b$$

$$\sigma_E: \mathcal{B} \times E \rightarrow \mathcal{E} \quad \neq 0$$

$$(b, e) \mapsto \sigma(b) + e$$

$$\sigma^{-1}(0) \simeq \text{zero} \left(\begin{array}{c} \sigma_E^{-1}(0) \rightarrow E \\ (b, e) \mapsto e \end{array} \right)$$

finite dim. reduction



$$U = \sigma_E^{-1}(0)$$

$$s: (b, e) \mapsto e$$

PHILOSOPHY: " $[\sigma^{-1}(0)]$ "

= Euler class of $\begin{array}{c} E \\ \downarrow \\ \sigma_E^{-1}(0) \end{array}$

$$= [(s+\tau)^{-1}(0)]$$