Analytic description of moduli spaces

Geometric Regularization

- "transverse gluing" — local surjectivity

Abstract Regularization

- Why?
- Fredholm stabilization
- "stabilized gluing"

$\boxed{\text{Geometric Regularization}}$

($\underline{\text{Ex}}$: Hamiltonian Floer without bubbling)

0.) Global Fredholm description of $\widetilde{\mathcal{M}} = \bar{\partial}_{J,H}^{-1}(0)$

1.) Equivariant Transversality

find J s.t. $\begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{B} = C^\infty(\mathbb{R}\times S^1, M) \end{matrix}$ $\bar{\partial}_{J,H} \pitchfork 0 \Rightarrow \bar{\partial}_{J,H}^{-1}(0) \subset \mathcal{B}$ smooth submanifold

2.) Quotient Theorem

$\mathbb{R} \circlearrowright \bar{\partial}_{J,H}^{-1}(0)$ smooth, proper, free $\Rightarrow \mathcal{M} = \frac{\bar{\partial}_{J,H}^{-1}(0)}{\mathbb{R}}$ smooth manifold

$= \bigcup_{k\geq 0} \mathcal{M}^k \quad \dim \mathcal{M}^k = k$

3.) Gromov Compactness & Gluing

$g: \mathcal{M}^0 \times_{\mathrm{Per}} \mathcal{M}^0 \times (R_1,\infty) \longrightarrow \mathcal{M}^1$

(1) injective

(2) locally surjective: $[u_i] \xrightarrow[i\to\infty]{\text{Gromov}} ([u_-],[u_+])$

$\Rightarrow \forall i \geq i_0 \; [u_i] \in \operatorname{im} g$

$\Rightarrow \overline{\mathcal{M}}^1 = \mathcal{M}^1 \cup_g \mathcal{M}^0 \times_{\mathrm{Per}} \mathcal{M}^0 \times (R_1,\infty]$ compact 1-mfd with $\partial \overline{\mathcal{M}}^1 \simeq \mathcal{M}^0 \times_{\mathrm{Per}} \mathcal{M}^0$

4.) Cobordism / Continuation map

for J_0 + J_1 construct $(CF, \partial_0) \simeq (CF, \partial_1)$ from $\mathcal{M}((J_t)_{t\in[0,1]})$

$\rightsquigarrow$ steps 1 - 3 twice more [see Salamon script]

local surjectivity: $[u_i] \xrightarrow[i\to\infty]{\text{Gromov}} ([u_-],[u_+]) \Rightarrow \forall i \geq i_0 \; [u_i] \in \operatorname{img}$

$$\Updownarrow$$

$$\left(\begin{array}{l} u_i(\cdot - S_i, \cdot) \longrightarrow u_- \\ u_i(\cdot + S_i, \cdot) \longrightarrow u_+ \\ E(u_-) + E(u_+) = E(u_i) = E \end{array} \quad \mathcal{C}^\infty_{loc}, \quad S_i \to \infty \right)$$

$$\int_{-k}^{k} |\partial_s u_-|^2 + \int_{-k}^{k} |\partial_s u_+|^2 \approx \int_{-k-S_i}^{k-S_i} |\partial_s u_i|^2 + \int_{-k+S_i}^{k+S_i} |\partial_s u_i|^2 \geq E - \varepsilon_k \quad \forall i \geq I_k$$

$$\Downarrow$$

$$u_i = \exp_{u_- \#_{S_i} u_+} \xi_i$$

$$\|\xi_i\|_{W^{1,p}} \leq C \left(\underbrace{\begin{array}{l} d_{(-k,k)}(u_i(\cdot - S_i), u_-) \\ + d_{(-k,k)}(u_i(\cdot + S_i), u_+) \end{array}}_{\text{fix } k,\; i\to\infty \;\to 0} \begin{array}{l} + d_{(k-S_i, -k+S_i)}(u_i, \gamma) \\ + d_{(-\infty,-k-S_i)}(u_i, \gamma_-) + d_{(-\infty,-k)}(\gamma_-, u_-) + d_{(-\infty,-k)}(\gamma, \exp_{u_+}(\beta \eta_+)) \\ + d_{(k+S_i,\infty)}(u_i, \gamma_+) + d_{(k,\infty)}(\gamma_+, u_+) + d_{(k,\infty)}(\gamma, \exp_{u_-}(\beta \eta_-)) \end{array} \right)$$

$$\underbrace{}_{\leq C \varepsilon_k}$$

$< \Delta/2$ by choice of k, $i \geq I_{k,\Delta}$

$$\Uparrow \text{ Newton iteration}$$

$$u_i(\cdot - \varepsilon \cdot, \cdot) = \exp_{u_- \#_{R_i} u_+} \xi$$

$$\xi \in \operatorname{im} D^*_{R_i}, \; \|\xi\|_{W^{1,p}} < \Delta$$

$$\text{IFT} \Uparrow \quad \exists\, r_i, \varepsilon_i: \; \xi_{r_i,\varepsilon_i} \in \operatorname{im} D^*_{S_i + r_i}$$

$$\Rightarrow u_i(\cdot + \varepsilon, \cdot) = \exp_{u_- \#_{S_i + r} u_+} \xi_{r,\varepsilon}$$

$$\|\xi_{r,\varepsilon}\| < \Delta \quad \forall |r|, |\varepsilon| \leq \delta_\Delta$$

IFT: $f(r,\varepsilon) := \begin{pmatrix} \langle \xi_{r,\varepsilon}, \alpha^-_r \rangle \\ \langle \xi_{r,\varepsilon}, \alpha^+_r \rangle \end{pmatrix} \quad \left(\operatorname{span}(\alpha^-_r, \alpha^+_r) = \ker D_{u_- \#_{S_i + r} u_+} \bar\partial_{J,H} \right)^\perp = \operatorname{im} D^*_{S_i + r}$

$$\xi_{r,\varepsilon} \approx u_i(\cdot + \varepsilon) - u_-(\cdot - S_i - r) - u_+(\cdot + S_i + r) \qquad \begin{array}{l} \alpha^-_r \approx \partial_s u_-(\cdot - S_i - r, \cdot) \\ \alpha^+_r \approx \partial_s u_+(\cdot + S_i + r, \cdot) \end{array} \qquad \langle \alpha^-_r, \alpha^+_r \rangle \approx 0$$

$$\Rightarrow \begin{array}{l} \partial_\varepsilon \xi_{r,\varepsilon}\big|_{\varepsilon, r = 0} \approx \partial_s u_i \approx \alpha^-_0 + \alpha^+_0 \\ \partial_r \xi_{r,\varepsilon}\big|_{\varepsilon, r = 0} \approx \partial_s u_-(\cdot - S_i) - \partial_s u_+(\cdot + S_i) = \alpha^-_0 - \alpha^+_0 \end{array}$$

$\xi_{r,\varepsilon} \approx 0$

$$\Rightarrow df(0,0) \approx \begin{pmatrix} \langle \alpha^- - \alpha^+, \alpha^- \rangle, & \langle \alpha^- + \alpha^+, \alpha^- \rangle \\ \langle \alpha^- - \alpha^+, \alpha^+ \rangle, & \langle \alpha^- + \alpha^+, \alpha^+ \rangle \end{pmatrix} \approx \begin{pmatrix} |\alpha^-|^2 & |\alpha^-|^2 \\ -|\alpha^+|^2 & |\alpha^+|^2 \end{pmatrix} \quad \text{invertible}$$

Abstract Regularization

Reason : regular J rarely exist

Tool : generalized regularization theorem for sections with $s^{-1}(0)$ compact

Approach : $\overline{\mathcal{M}} = s^{-1}(0)$

0.) quotient & compactify / glue $\quad \overline{\mathcal{M}} = \widetilde{\mathcal{M}}/_{\mathrm{Aut}} \cup \mathcal{M}^{\mathrm{broken}}$

1.) local Fredholm descriptions $\quad \overline{\mathcal{M}} = \dot{\bigcup} F_i \qquad$ Q: near $\mathcal{M}^{\mathrm{broken}}$?

$$\begin{array}{c} \mathcal{E}_i \\ \downarrow \uparrow s_i \\ \mathcal{B}_i \end{array} \quad \text{Fredholm section} \quad , \quad s_i^{-1}(0) \xrightarrow[\text{homeom}]{\sim} F_i \underset{\text{open}}{\subset} \overline{\mathcal{M}}$$

1'.) transition information $\quad \overline{\mathcal{M}} = {\sqcup F_i}/_{\sim} \quad = \text{"} s^{-1}(0) \text{"}$

2.) regularization theorem : $\exists \{\nu\}$: $\quad s + \nu \pitchfork 0 \quad$ (and $(s+\nu)^{-1}(0)$ mfd)

$$\overline{\mathcal{M}}^{\nu} := (s+\nu)^{-1}(0) \quad \text{compact} \quad , \quad \partial \overline{\mathcal{M}}^{\nu} \simeq \overline{\mathcal{M}}^{\nu} \times \overline{\mathcal{M}}^{\nu}$$

unique up to cobordism

local Fredholm description

$$\begin{array}{c} \mathcal{E} \\ \downarrow \uparrow \sigma \\ \mathcal{B} \end{array}$$ Fredholm section, $\sigma^{-1}(0) \hookrightarrow \overline{\mathcal{M}}$ local homeom

finite symmetry group

obstruction "bundle" — **stabilization** — **finite dim. reduction**

$$\begin{array}{c} U \quad \mathcal{E}_b / \mathrm{im}\, D_b\sigma \\ b\in\sigma^{-1}(0) \\ \downarrow \\ \sigma^{-1}(0) \end{array}$$

$\rightsquigarrow$ $E \subset \mathcal{E}|_{\mathrm{nbhd}(\sigma^{-1}(0))}$

$E_b + \mathrm{im}\, D_b\sigma = \mathcal{E}_b$

$\sigma_E : \mathcal{B}\times E \longrightarrow \mathcal{E}$ $\pitchfork 0$

$(b,e) \longmapsto \sigma(b)+e$

$\sigma^{-1}(0) \simeq \mathrm{zero}\left(\begin{array}{c} \sigma_E^{-1}(0) \to E \\ (b,e) \mapsto e \end{array} \right)$

$\Rightarrow$ $$\begin{array}{c} U\times E \\ \downarrow \uparrow s \\ U \end{array} \quad \circlearrowleft \Gamma$$ finite

$s^{-1}(0)/\Gamma \hookrightarrow \mathcal{M}$ local homeom.

$U = \sigma_E^{-1}(0)$

$s : (b,e) \longmapsto e$

PHILOSOPHY: "$[\sigma^{-1}(0)]$" $=$ Euler class of $\begin{array}{c} E \\ \downarrow \\ \sigma_E^{-1}(0) \end{array}$ $= [(s+\nu)^{-1}(0)]$