Analytic description of moduli spaces – Gluing

References:

McDuff - Salamon, J-hol. curves and Symplectic Topology §10, A3

Matthias Schwarz, PHD thesis http://www.math.uni-leipzig.de/~schwarz/diss.pdf

Audin - Damian, Morse Theory and Floer Homology §9
(new – translated – at Springer)

Gluing in Hamiltonian Floer Theory (transverse index 0 case)

"expected dim = Fred index -1"

$u_\pm : \mathbb{R}\times S^1 \to M$, $\quad \bar\partial_{J,H} u_\pm = 0$, $\quad \lim\limits_{s\to+\infty} u_-(s,\cdot) = \gamma = \lim\limits_{s\to-\infty} u_+(s,\cdot)$

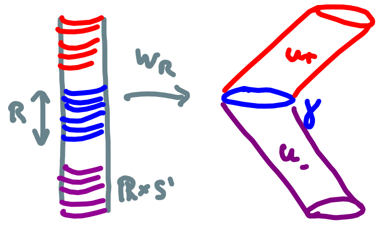
($\pitchfork$, isolated broken trajectories) $\qquad$ exp. decay: $|\partial_s u_\pm(s,t)| \le C e^{\delta|s|}$; $\delta>0$

$D_\pm := D_{u_\pm}\bar\partial_{J,H} : W^{1,p}(\mathbb{R}\times S^1, u_\pm^* TM) \to L^p(\mathbb{R}\times S^1, u_\pm^* TM)$ $\quad$ surjective

$\ker D_\pm = \mathbb{R}\,\partial_s u_\pm$

<u>pregluing</u>

$$w_R = \#_R(u_-,u_+) = \begin{cases} u_-(\cdot + R, \cdot) & \text{on } (-\infty, -\frac{R}{2}-1]\times S^1 \\ \text{interpolation} & \\ \gamma & \text{on } [-\frac{R}{2}, \frac{R}{2}]\times S^1 \\ \text{interpolation} & \\ u_+(\cdot - R, \cdot) & \text{on } [\frac{R}{2}+1, \infty)\times S^1 \end{cases}$$



Newton iteration to solve $f_R(\xi) = 0$

$$f_R : \begin{array}{ccc} W^{1,p}(\mathbb{R}\times S^1, w_R^* TM) \supset \{\|\xi\| < \Delta\} & \longrightarrow & L^p(\mathbb{R}\times S^1, w_R^* TM) \\ \xi & \longmapsto & \underset{\text{along } \exp_{w_R}(\tau\xi)}{\text{par.transp.}} \left(\bar\partial_{J,H} \exp_{w_R}(\xi)\right) \end{array}$$

<u>Claim</u>: $\forall R \geq R_1 \ \forall (u_-, u_+) \in \mathcal{M} \times \mathcal{M}$

(a) $\| f_R(0) \|_{L^p} \leq C_0 e^{-\delta R}$ "almost sol$^{\text{n}}$"

(b) $\| \xi \|_{W^{1,p}} \leq C_1 \| df_R(0) \xi \|_{L^p} \quad \forall \xi \in \operatorname{im} D_R^*$ "right inverse"

(c) $\| f_R(\xi + \hat{\xi}) - f_R(\xi) - df_R(\xi)\hat{\xi} \|_{L^p} \leq C_2 \| \hat{\xi} \|^2_{W^{1,p}} \quad \forall \|\xi\|_{W^{1,p}}, \|\hat{\xi}\|_{W^{1,p}} \leq 1$ $\quad f_R$ is $\mathcal{C}^1$

$\Downarrow$ Newton iteration

$$\forall R \geq \delta^{-1} \ln(8 C_0 C_1^2 C_2) \quad \exists! \, \xi_R \in \underline{\operatorname{im} D_R^*} \subset W^{1,p}(w_R^* TM) : \begin{cases} \bar{\partial}_{J,H} \exp_{w_R}(\xi_R) = 0 \\ \|\xi\|_{W^{1,p}} < \Delta \end{cases}$$

in fact $\|\xi_R\|_{W^{1,p}} \leq C' e^{-\delta R}$

<u>Gluing Map</u> (transverse case)

$$g : \bigcup_{\gamma \text{ per. orbit}} \mathcal{M}^0(\gamma_-, \gamma) \times \mathcal{M}^0(\gamma, \gamma_+) \times (R_1, \infty] \longrightarrow \overline{\mathcal{M}}^1(\gamma_-, \gamma_+) = \bar{\partial}_J^{-1}(0)/\mathbb{R} \cup \{\text{broken}\}$$

$$([u_-], [u_+], R) \longmapsto \begin{cases} [\exp_{w_R}(\xi_R)] & ; R < \infty \\ ([u_-], [u_+]) & ; R = \infty \end{cases}$$

Gluing Map (transverse case)

$$g : \bigcup_{\gamma \text{ per. orbit}} \mathcal{M}^0(\gamma_-,\gamma) \times \mathcal{M}^0(\gamma,\gamma_+) \times (R_1,\infty] \longrightarrow \overline{\mathcal{M}}^1(\gamma_-,\gamma_+)$$

$$([u_-],[u_+],R) \longmapsto \begin{cases} [\exp_{\#_R(u_-,u_+)}(\zeta_R)] & ; R<\infty \\ ([u_-],[u_+]) & ; R=\infty \end{cases}$$

Claim

(0) g well defined ✓ & continuous

pick representatives $u=[u]\in\mathcal{U}$ for all isolated points $\mathcal{U}\in\mathcal{M}^0$

↳ in particular

$g([u_-],[u_+],R) \xrightarrow[R\to\infty]{} ([u_-],[u_+])$ in Gromov topology

(1) g injective

(2) g locally surjective $\forall [u_-],[u_+]\ \exists \text{nbhd}([u_-],[u_+]) \subset \overline{\mathcal{M}}^1(\gamma_-,\gamma_+)$:

$[v]\in\text{nbhd} \Rightarrow \exists R > R_1 : [v] = g([u_-],[u_+],R)$

$\Leftrightarrow \exists R,s : v(\cdot+s,\cdot) = \exp_{w_R}(\zeta)$, $\begin{cases} \zeta\in\text{im}\,Q_R \\ \|\zeta\|<\Delta \end{cases}$

Newton iteration implies "uniqueness"

$$\left.\begin{aligned} &\bar\partial_{J,H} u = 0,\ u=\exp_{w_R}(\zeta) \\ &\zeta\in\text{im}\,Q_R,\ \|\zeta\|<\Delta \end{aligned}\right\} \Rightarrow \zeta=\zeta_R,\ u = g([u_-],[u_+],R)$$

(0) <u>continuity</u> • at $R<\infty$ follows from continuity of f_R, df_R wrt R

• at $R=\infty$, given $\varepsilon>0$ find R_ε s.t. $\forall r>R_\varepsilon$

$$\underset{\underset{[\exp_{w_r}(\zeta_r)]}{\|}}{g(r)} \in \begin{matrix}\varepsilon\text{-nbhd of}\\ ([u_-],[u_+])\end{matrix} = \Big\{\gamma\in\bar{\mathcal{M}}(\gamma_-,\gamma_+)\,\Big|\,\exists V\in\mathcal{V},\ R>1/\varepsilon : \underbrace{d_{W^{1,p}}(v,\#_R(u_-,u_+))}_{\approx\|\zeta_r\|}<\varepsilon\Big\}$$

$\|\zeta_r\|\le C\|f_r(0)\|<Ce^{-\delta r}$ $\Rightarrow$ pick R_ε s.t. $Ce^{-\delta R_\varepsilon}\le\varepsilon$

---

(1) <u>injectivity</u> $\boxed{\text{SKETCH}}$ $\not\downarrow$-assumption: $\exists\,([u_-^i],[u_+^i],R_i)\neq([v_-^i],[v_+^i],S_i)$:

$$g([u_-^i],[u_+^i],R_i)=g([v_-^i],[v_+^i],S_i)\quad,\quad R_i,S_i\to\infty$$

* w.l.o.g. $u_\pm^i=u_\pm$, $v_\pm^i=v_\pm$ (for subsequence since $\mathcal{M}^0$ is finite

* $g([u_-],[u_+],R_i)\longrightarrow([u_-],[u_+])$
  $\quad\|$
  $g([v_-],[v_+],S_i)\longrightarrow([v_-],[v_+])$

$\Rightarrow [u_\pm]=[v_\pm]$

Hausdorff property of Gromov topology

$$\rho([u_-],[u_+],R_i) = \rho([u_-],[u_+],S_i) \qquad , R_i \neq S_i \to \infty$$

$$\Rightarrow \quad \underset{\Downarrow}{\exp_{w_{R_i}}(\zeta_{R_i})} = \underset{\Downarrow}{\exp_{w_{S_i}}(\eta_{S_i})}(\cdot + T_i, \cdot) \qquad \text{for some } T_i \in \mathbb{R}$$

$$\bullet \Rightarrow \begin{cases} u_-(\cdot + R_i, \cdot) \approx u_-(\cdot + S_i + T_i, \cdot) & \text{on } (-\infty, \min\{-\frac{R_i}{2}, -\frac{S_i}{2} - T_i\}) \\ u_+(\cdot - R_i, \cdot) \approx u_+(\cdot - S_i + T_i, \cdot) & \text{on } (\max\{\frac{R_i}{2}, \frac{S_i}{2} - T_i\}, \infty) \end{cases}$$

$$\Rightarrow \begin{array}{ll} u_- \approx u_-(\cdot + S_i - R_i + T_i, \cdot) & \text{on } (-\infty, \min\{\frac{R_i}{2}, R_i - \frac{S_i}{2} - T_i\}) \\ u_+ \approx u_+(\cdot - (S_i - R_i - T_i), \cdot) & \text{on } (\max\{-\frac{R_i}{2}, -R_i + \frac{S_i}{2} - T_i\}, \infty) \end{array}$$

* finite energy $\Rightarrow u_\pm$ "not close to periodic"

$$\Rightarrow \begin{array}{l} S_i - R_i + T_i \to 0 \text{ or } \frac{S_i}{2} - R_i + T_i \to \infty \\ S_i - R_i - T_i \to 0 \text{ or } \frac{S_i}{2} - R_i - T_i \to \infty \end{array}$$

not so useful

added after lecture

$$\bullet \Rightarrow 0 \approx \text{Energy}\left(w_{S_i}(\cdot + T_i, \cdot)\big|_{[-\frac{R_i}{2}, \frac{R_i}{2}]}\right) \geq \text{Energy}\left(u_-\big|_{[S_i - \frac{R_i}{2} + T_i, \min\{S_i + \frac{R_i}{2} + T_i, \frac{S_i}{2}\}]}\right)$$
$$+ \text{Energy}\left(u_+\big|_{[\min\{-\frac{S_i}{2}, -S_i - \frac{R_i}{2} + T_i\}, -S_i + \frac{R_i}{2} + T_i]}\right)$$

$$\Rightarrow \begin{cases} \frac{S_i}{2} - \frac{R_i}{2} + T_i \to 0 & \text{or} \quad S_i + \frac{R_i}{2} + T_i \to -\infty \\ \frac{S_i}{2} - \frac{R_i}{2} - T_i \to 0 & \text{or} \quad -S_i - \frac{R_i}{2} + T_i \to \infty \end{cases}$$

interval length $\to 0$ interval $\to \pm\infty$

$$\Rightarrow \begin{pmatrix} S_i + \frac{R_i}{2} + T_i \to -\infty \\ \frac{S_i}{2} - \frac{R_i}{2} - T_i \to 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{S_i}{2} - \frac{R_i}{2} + T_i \to 0 \\ -S_i - \frac{R_i}{2} + T_i \to \infty \end{pmatrix} \text{ or } \begin{pmatrix} S_i + \frac{R_i}{2} + T_i \to -\infty \\ -S_i - \frac{R_i}{2} + T_i \to \infty \end{pmatrix} \text{ or } \begin{pmatrix} \frac{S_i}{2} - \frac{R_i}{2} + T_i \to 0 \\ \frac{S_i}{2} - \frac{R_i}{2} - T_i \to 0 \end{pmatrix}$$

$\Rightarrow \frac{3S_i}{2} \to -\infty$ ↯ $\Rightarrow -\frac{3S_i}{2} \to \infty$ ↯ $\Rightarrow T_i \to -\infty \;\&\; T_i \to \infty$ ↯ $\Downarrow$ $S_i - R_i \to 0$, $T_i \to 0$

$\Rightarrow$ REMAINS TO SHOW: <u>LOCAL INJECTIVITY</u>

$u_\pm$ fixed, $R \neq S$, $|R - S| < \varepsilon \Rightarrow \rho(R) \neq \rho(S)$

This follows from

$$\begin{array}{c} \partial_R \rho \\ \| \\ \partial_R w_R + \partial_R \zeta_R \end{array} \qquad \rho(R) = \exp_{w_R}(\zeta_R) \sim w_R + \zeta_R$$

$$0 \neq \begin{cases} \partial_s u_- \\ -\partial_s u_+ \end{cases} = \partial_R w_R, \qquad \partial_R \zeta_R \xrightarrow{R \to \infty} 0 \quad \text{(see later)}$$

$$\partial_R g = \partial_R\big(\mathrm{Exp}(w_R, \zeta_R)\big) \approx \partial_R (w_R + \zeta_R)$$

① ②

$$= \underbrace{\partial_1 \mathrm{Exp}(w_R, \underset{\approx 0}{\zeta_R})}_{\approx id} \underbrace{\partial_R w_R}_{\neq 0} + \underbrace{\partial_2 \mathrm{Exp}(w_R, \underset{\approx 0}{\zeta_R})}_{id} \partial_R \zeta_R$$

small?

$$\boxed{f_R(\zeta_R) = 0}$$

$$\Rightarrow \quad Df_R(\zeta_R)\, \partial_R \zeta_R = -\partial_R f_R(\zeta_R) = \dots \xrightarrow[R\to\infty]{} 0$$

know: $\|\eta\| \leq c \|Df_R(0)\eta\|$ for $\eta \in \mathrm{im}\, Q_R$ (complement to $\ker Df_R$)

$\zeta_R \in \mathrm{im}\, Q_R$

If $\partial_R \zeta_R \in \mathrm{im}\, Q_R$, then would imply $\partial_R \zeta_R \to 0$.